L01 — Complex Numbers and Complex Plane

Engineering Mathematics I

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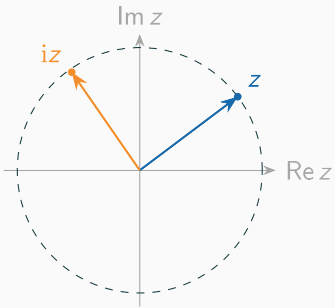
Road map

- Arithmetic and field structure
- The Gauss-Argand plane (or complex plane)
- Polar/exponential form
- Powers, roots, and polynomial roots
- The Riemann sphere



Complex Numbers and Arithmetic

Why invent i?

The equation $x^2 = -1$ has no solution in $\mathbb{R}$. We enlarge the number system by defining

$$\mathrm{i}^2 = -1, \qquad \sqrt{-1} = \mathrm{i}.$$

$$\boxed{\mathbb{N}} \longrightarrow \boxed{\mathbb{Z}} \longrightarrow \boxed{\mathbb{Q}} \longrightarrow \boxed{\mathbb{R}} \longrightarrow \boxed{\mathbb{C}}$$

$x^2 + 1 = 0$ is solvable

Definition

A complex number has the form

$$z = a + \mathrm{i}\, b, \qquad a, b \in \mathbb{R}.$$

- $a = \mathrm{Re}(z)$ is the *real part*,
- $b = \mathrm{Im}(z)$ is the *imaginary part*.

The imaginary unit satisfies

$$\mathrm{i}^2 = -1.$$

Addition, Subtraction, and Multiplication

Complex numbers are added componentwise:

$$(a + \mathrm{i}b) + (c + \mathrm{i}d) = (a + c) + \mathrm{i}(b + d).$$

Subtraction is defined as:

$$(a + \mathrm{i}b) - (c + \mathrm{i}d) = (a - c) + \mathrm{i}(b - d).$$

Multiplication follows from the distributive law and $\mathrm{i}^2 = -1$:

$$(a + \mathrm{i}b)(c + \mathrm{i}d) = (ac - bd) + \mathrm{i}(ad + bc).$$

Conjugation makes division possible

If $z = x + \mathrm{i}y$, its *complex conjugate* is

$$\bar{z} = x - \mathrm{i}y.$$

Geometrically, conjugation is reflection across the real axis (will be shown later).

To divide, multiply numerator and denominator by the conjugate of the denominator. For $w = c + \mathrm{i}d \neq 0$,

$$\frac{z}{w} = \frac{(x+\mathrm{i}y)(c-\mathrm{i}d)}{c^2+d^2} = \frac{z\overline{w}}{w\overline{w}} = \frac{a+\mathrm{i}b}{c+\mathrm{i}d} = \frac{(a+\mathrm{i}b)(c-\mathrm{i}d)}{c^2+d^2}.$$

In particular

$$\frac{1}{z} = \frac{1}{a+\mathrm{i}b} = \frac{a}{a^2+b^2} - \mathrm{i}\,\frac{b}{a^2+b^2} = \frac{\bar{z}}{|z|^2}.$$

Field structure of $\mathbb{C}$

With the preceding operations, $\mathbb{C}$ is a field. For $z_1, z_2, z_3 \in \mathbb{C}$:

- closure under $+$ and $\cdot$;
- commutativity;
- associativity;
- identities 0 and 1;
- additive inverse $-z$;
- multiplicative inverse z^{-1} for $z \neq 0$;
- distributivity:

$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3.$$

Unlike $\mathbb{R}$, the complex field has no order compatible with its arithmetic. [1]

[1] There is no way to define $>$ on $\mathbb{C}$ such that $z_1 > z_2 \Rightarrow z_1 + z_3 > z_2 + z_3$ and $z_1 > 0, z_2 > 0 \Rightarrow z_1 z_2 > 0$.

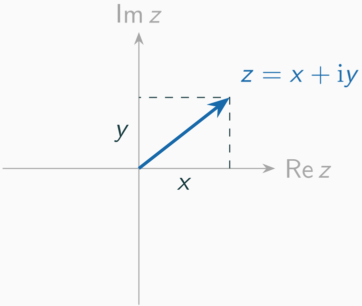
Complex Plane

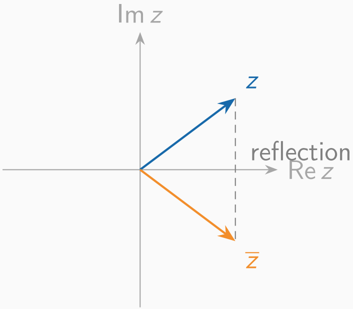
Geometric representation of complex numbers

Every complex number $z = x + \mathrm{i}y$ may be identified with

- the point (x, y) in the Euclidean plane $\mathbb{R}^2$, or
- the vector from the origin to that point.

The set of all such points is called the *complex plane*, or the (Gauss-)Argand plane.





Conjugation is reflection across the real axis.

Revisit addition, subtraction and multiplication

For $z_1 = x_1 + \mathrm{i}y_1$ and $z_2 = x_2 + \mathrm{i}y_2$, addition and subtraction act componentwise; multiplication uses distributivity and $\mathrm{i}^2 = -1$.

Addition

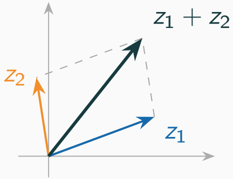
$$z_1 + z_2 = (x_1 + x_2) + \mathrm{i}(y_1 + y_2)$$

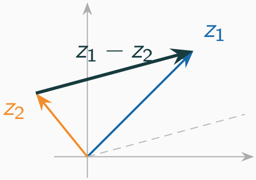
Subtraction

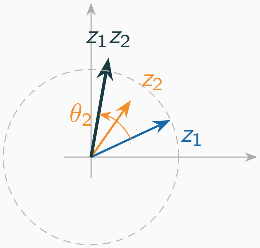
$$z_1 - z_2 = (x_1 - x_2) + \mathrm{i}(y_1 - y_2)$$

Multiplication

$$\begin{aligned} z_1 z_2 &= (x_1x_2 - y_1y_2) + \mathrm{i}(x_1y_2 + x_2y_1) \\ &= r_1 r_2 \mathrm{e}^{\mathrm{i}(\theta_1+\theta_2)}, \text{(will be shown later)}. \end{aligned}$$







Each operation has a distinct geometry: Addition uses a parallelogram; subtraction measures displacement; multiplication rotates and scales.

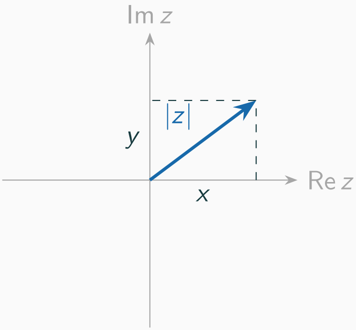
Modulus: algebra meets geometry

The *modulus* (or absolute value) of z is the non-negative real number

$$|z| = \sqrt{x^2 + y^2} = \sqrt{z\,\bar{z}}.$$

Key properties:

- $|z| \geq 0$ and $|z| = 0 \Leftrightarrow z = 0$,
- $|\bar{z}| = |z|$,
- $|zw| = |z|\,|w|$,
- $\left|\dfrac{z}{w}\right| = \dfrac{|z|}{|w|}$ $(w \neq 0)$.

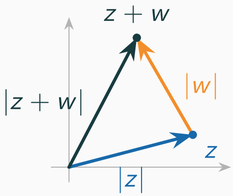


Distance and the triangle inequality

Distance in the complex plane is $d(z, w) = |z - w|$.

Triangle inequality

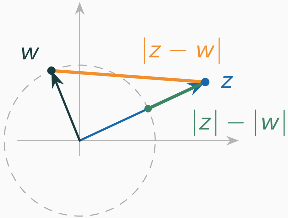
$$|z + w| \leq |z| + |w|$$



The direct side is no longer than the two-step path.

Reverse triangle inequality

$$\big||z| - |w|\big| \leq |z - w|$$

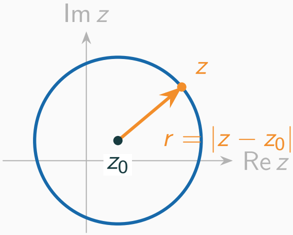


The chord is at least the difference of the radii.

Loci from modulus equations

Circle

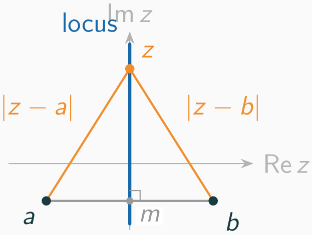
$$|z - z_0| = r$$



Every point on the locus is exactly distance r from z_0.

Perpendicular bisector

$$|z - a| = |z - b|$$

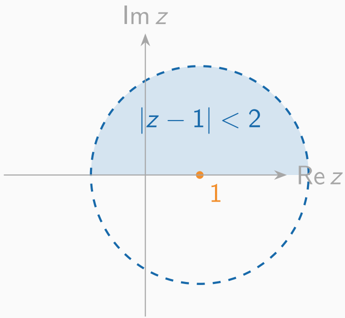


The locus passes through the midpoint m and is perpendicular to ab.

Regions in the complex plane

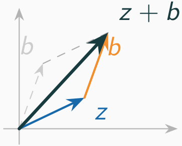
Example: sketch $|z-1|<2$ and $\operatorname{Im} z>0$.

- $|z-1|<2$: interior of a disk centered at 1;
- $\operatorname{Im} z>0$: upper half-plane;
- both conditions: their intersection.

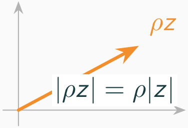


Geometric transformations

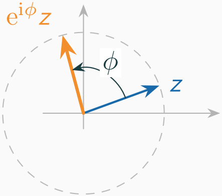
Translation $z \mapsto z + b$



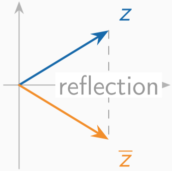
Dilation $z \mapsto \rho z$



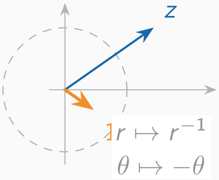
Rotation $z \mapsto \mathrm{e}^{\mathrm{i}\phi} z$



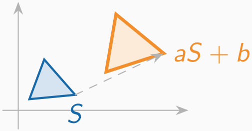
Conjugation $z \mapsto \overline{z}$



Reciprocal $z \mapsto 1/z$



Affine map $z \mapsto az + b$



Polar Coordinates and Argument

Polar form and Euler's formula

Every $z \neq 0$ can be written

$$z = r(\cos\theta + \mathrm{i}\sin\theta) = r\mathrm{e}^{\mathrm{i}\theta}, \qquad r = |z| > 0,$$

where Euler's formula connects the exponential and trigonometric representations:

$$\boxed{\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta}$$

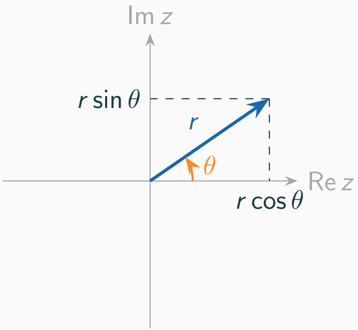
Polar to Cartesian conversion:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r = \sqrt{x^2 + y^2}.$$

The angle θ is an *argument* of z. Euler's formula packages trigonometry into exponential algebra:

$$\mathrm{e}^{\mathrm{i}(\theta_1+\theta_2)} = \mathrm{e}^{\mathrm{i}\theta_1}\mathrm{e}^{\mathrm{i}\theta_2}.$$

Special case: $\mathrm{e}^{\mathrm{i}\pi} + 1 = 0$.

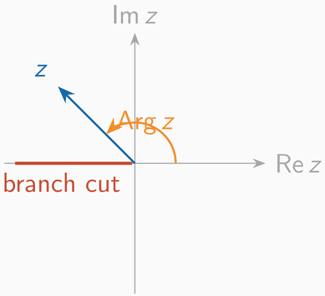


Principal argument

All arguments differ by an integer multiple of 2π:

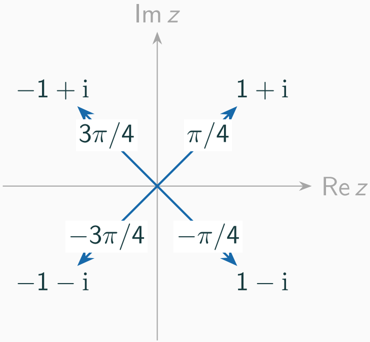
$$\arg z = \operatorname{Arg} z + 2k\pi, \qquad k \in \mathbb{Z},$$

with the principal value convention $\operatorname{Arg} z \in (-\pi, \pi]$.



Finding $\operatorname{Arg} z$: use the quadrant

A reliable formula is $\theta = \arctan(y, x)$.



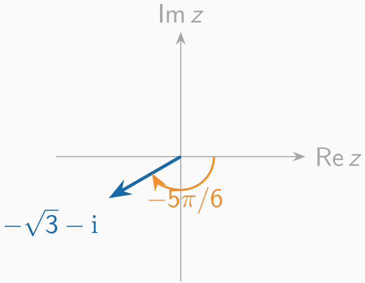
Example — algebraic to polar form

For $z = -\sqrt{3} - \mathrm{i}$,

$$r = \sqrt{3+1} = 2, \qquad \operatorname{Arg} z = -\frac{5\pi}{6}.$$

Hence

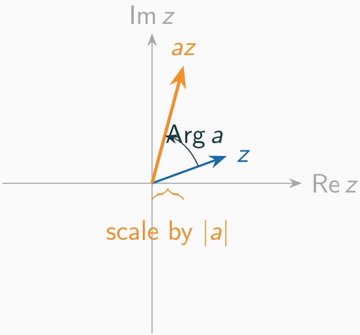
$$z = 2\left(\cos\left(-\frac{5\pi}{6}\right) + \mathrm{i}\sin\left(-\frac{5\pi}{6}\right)\right) = 2\mathrm{e}^{-5\pi\mathrm{i}/6}.$$



Multiplication = dilation + rotation

If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}, \qquad \frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}.$$



EE: phasors turn sinusoidal steady state into geometry

A sinusoid can be encoded by one complex amplitude:

$$v(t) = \mathrm{Re}\{\underline{V}\mathrm{e}^{\mathrm{i}\omega t}\},$$
$$\underline{V} = V_m \mathrm{e}^{\mathrm{i}\phi_V}.$$

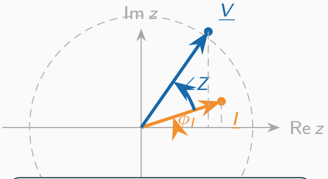
For a linear AC element or network,

$$\underline{V} = Z\underline{I}, \qquad Z = R + \mathrm{i}X.$$

Thus the same multiplication rule just learned gives the voltage gain and phase shift.

Interpretation

$|Z|$ scales the current phasor and Arg Z rotates it.



$\underline{V} = Z\underline{I}$

$|\underline{V}| = |Z|\,|\underline{I}|, \quad \phi_V - \phi_I = \mathrm{Arg}\,Z$

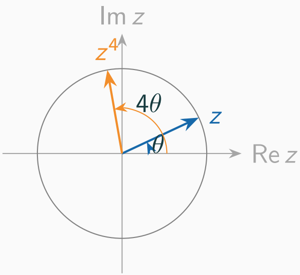
Powers, Roots, and Polynomials

De Moivre's formula

For every integer n,

$$(\cos\theta + \mathrm{i}\sin\theta)^n = \cos(n\theta) + \mathrm{i}\sin(n\theta),$$

or equivalently $(\mathrm{e}^{\mathrm{i}\theta})^n = \mathrm{e}^{\mathrm{i}n\theta}$.

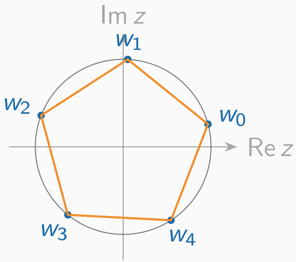


n-th roots of a complex number

For $z = r\mathrm{e}^{\mathrm{i}\theta} \neq 0$, the n distinct solutions of $w^n = z$ are

$$w_k = r^{1/n}\mathrm{e}^{\mathrm{i}(\theta+2k\pi)/n}, \qquad k = 0, 1, \dots, n-1.$$

They lie on a circle of radius $r^{1/n}$, separated by angle $2\pi/n$.



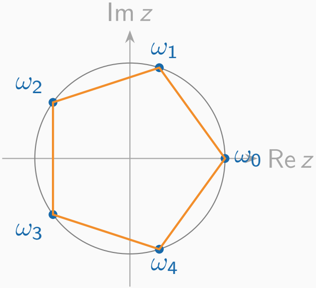
Roots of unity

The solutions of $w^n = 1$ are

$$\omega_k = \mathrm{e}^{2\pi \mathrm{i} k/n}, \qquad k = 0, \dots, n-1.$$

For $n = 5$, the roots are vertices of a regular pentagon. Their sum is zero:

$$1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0.$$



- The sum of the roots of unity is zero.
- This follows from the fact that the sum of the roots of a monic polynomial is equal to the negative of the coefficient of the second-highest degree term.

Why the root sum is zero

Let

$$p(z) = z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_0$$

be a monic polynomial of degree n, and suppose

$$p(z) = (z - r_1)(z - r_2)\cdots(z - r_n).$$

When you expand the product, the coefficient of z^{n-1} comes from choosing one factor $(-r_j)$ and the z from every other factor. Thus

$$z^{n-1} \text{ coefficient } = -(r_1 + r_2 + \cdots + r_n).$$

So the sum of the roots satisfies

$$r_1 + r_2 + \cdots + r_n = -a_{n-1}.$$

For the polynomial $p(z) = z^n - 1$, the coefficient of z^{n-1} is 0, which gives

$$\omega_0 + \omega_1 + \cdots + \omega_{n-1} = 0.$$

This is exactly why the nth roots of unity add up to zero.

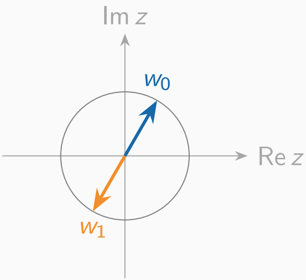
Example — square roots

Find w such that $w^2 = -1 + \sqrt{3}\,\mathrm{i}$.

$$z = 2\mathrm{e}^{2\pi\mathrm{i}/3} \quad\Longrightarrow\quad w_k = \sqrt{2}\,\mathrm{e}^{\mathrm{i}(\pi/3 + k\pi)}, \quad k = 0, 1.$$

Thus

$$w_0 = \frac{\sqrt{2}}{2}(1 + \sqrt{3}\,\mathrm{i}), \qquad w_1 = -w_0.$$



Fundamental theorem of algebra

Theorem

Every nonconstant polynomial with complex coefficients has a root in $\mathbb{C}$.

Consequently, a degree-n polynomial factors as

$$p(z) = a_n(z - z_1)(z - z_2)\cdots(z - z_n),$$

where roots are counted with multiplicity.

Real coefficients

Nonreal roots occur in conjugate pairs: if p has real coefficients and $p(z_0) = 0$, then $p(\overline{z_0}) = 0$.

Liouville's argument

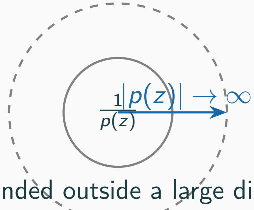
Suppose p is a nonconstant polynomial with no zeros in $\mathbb{C}$. Then

$$f(z) = \frac{1}{p(z)}$$

is an entire function. Since p has degree $n \geq 1$,

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, \quad |p(z)| \to \infty \text{ as } |z| \to \infty.$$

Therefore $f(z) \to 0$ as $|z| \to \infty$, so f is bounded on all of $\mathbb{C}$. By Liouville's theorem, f must be constant. But then $p = 1/f$ would also be constant, contradicting our assumption. Hence p must have at least one zero.

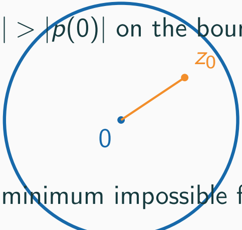


Minimum-modulus argument

Assume a nonconstant polynomial p has no zeros. Then p is analytic and nonzero on all of $\mathbb{C}$. Choose R so large that

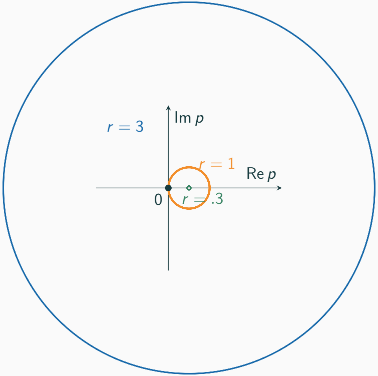
$$|p(z)| > |p(0)| \quad \text{for all } |z| = R.$$

Since $|p|$ is continuous on the closed disk $\overline{D(0,R)}$, it attains a minimum at some point z_0 with $|z_0| < R$. Because p has no zeros, this minimum value is positive. But the minimum-modulus principle says: if a nonzero analytic function has an interior minimum of its modulus, then the function is constant. So p would have to be constant, contradicting the hypothesis. Therefore a nonconstant polynomial must vanish somewhere in $\mathbb{C}$.



A geometric glimpse of the theorem

For $p(z) = z^2 + 1$ and $z = r\mathrm{e}^{\mathrm{i}\theta}$, the curve $p(r\mathrm{e}^{\mathrm{i}\theta})$ changes as r varies.

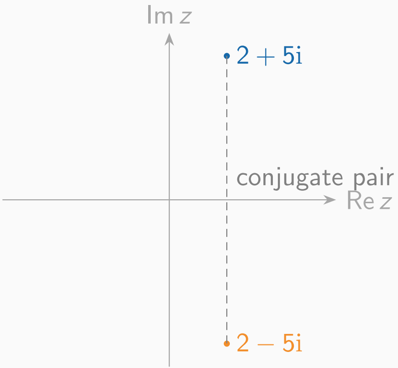


At $r = 1$, the image passes through 0: indeed $z = \pm\mathrm{i}$ are roots.

Polynomial example

Solve $x^2 - 4x + 29 = 0$:

$$x = \frac{4 \pm \sqrt{16 - 116}}{2} = \frac{4 \pm 10\mathrm{i}}{2} = 2 \pm 5\mathrm{i}.$$



Stereographic Projection

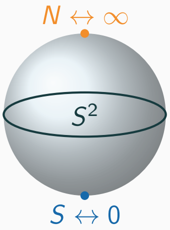
The extended complex plane

Add one point at infinity:

$$\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}.$$

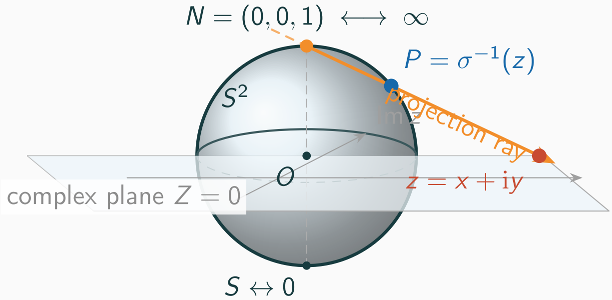
Stereographic projection identifies $\widehat{\mathbb{C}}$ with the unit sphere

$$S^2 = \{(X, Y, Z) \in \mathbb{R}^3 : X^2 + Y^2 + Z^2 = 1\}.$$



Stereographic projection: construction

Draw the line from the north pole N through $P \in S^2$; its intersection with the plane $Z = 0$ is z.



Projection formulas

For $z = x + \mathrm{i}y$, inverse stereographic projection gives

$$X = \frac{2x}{1+|z|^2}, \qquad Y = \frac{2y}{1+|z|^2}, \qquad Z = \frac{|z|^2-1}{1+|z|^2}.$$

Conversely, for $P = (X, Y, Z) \neq N$,

$$z = \frac{X + \mathrm{i}Y}{1 - Z}.$$

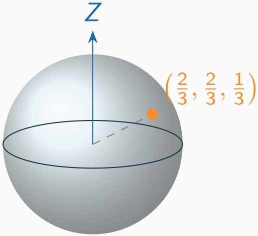
$$\begin{tikzcd} z = x + \mathrm{i}y \in \mathbb{C} \arrow[r, bend left, "\text{inverse projection}"] & P = (X, Y, Z) \in S^2 \arrow[l, bend left, "\text{projection}"] \end{tikzcd}$$

Worked example on the Riemann sphere

For $z = 1 + \mathrm{i}$, we have $|z|^2 = 2$, hence

$$(X, Y, Z) = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right).$$

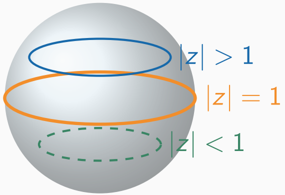
Because $Z > 0$, the point lies in the upper hemisphere.



Geometry under stereographic projection

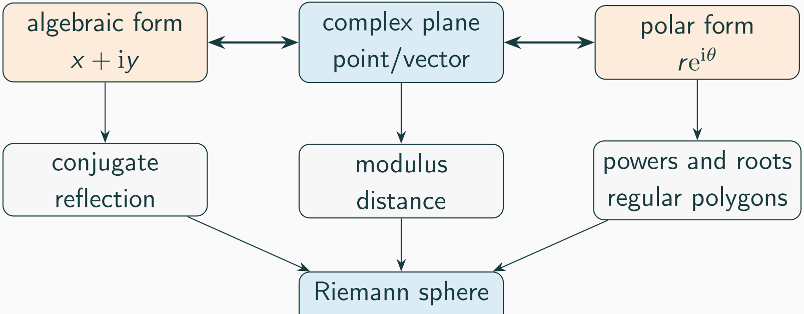
- $|z| < 1$ maps to the southern hemisphere;
- $|z| = 1$ maps to the equator;
- $|z| > 1$ maps to the northern hemisphere;
- circles through N become lines in $\mathbb{C}$;
- other circles become circles;
- angles are preserved (conformality).

See Wolfram Notebook demonstrations.



Summary: What we have learned & Concept map

- Complex arithmetic, inverses, and the field structure of $\mathbb{C}$.
- Conjugate, modulus, distance, and triangle inequalities.
- Geometric interpretation in the Gauss–Argand plane and common loci.
- Polar/exponential form and multiplication as scaling plus rotation.
- De Moivre's formula, n-th roots, roots of unity, and polynomial roots.
- Stereographic projection and the Riemann sphere $\widehat{\mathbb{C}}$.



Looking ahead

Next lectures will treat

- complex functions and mappings,
- visulatization,
- limits, continuity and complex differentiability,
- the Cauchy–Riemann equations.

Next » Lecture 02 — Limits, Continuity, Derivatives and Analyticity of Complex Functions