

L02 — Complex Functions

Limits, Continuity, Derivatives and Analyticity

Prof. Tianyu DONG & Jiawei WANG

Fall 2026

School of Electrical Engineering, Xi'an Jiaotong University

Road map

- Complex functions and their geometry
- Limits and continuity
- Complex differentiation
- Analyticity and the Cauchy–Riemann equations
- Harmonic consequences

L01 bridge: geometry of points becomes geometry of maps

Lecture 01 taught us to think of complex numbers as vectors and points in the plane:

$$z = re^{i\theta}, \quad |z| = r, \quad \text{Arg } z = \theta.$$

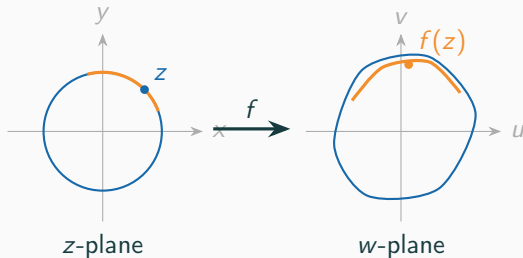
Now a complex function becomes a transformation of the plane:

$$w = f(z).$$

The same geometric language remains useful:

- modulus tells us how lengths are scaled (modulus measures stretching),
- argument tells us how directions are (rotated argument measures turning),
- curves and regions tell us how shapes are deformed (images of curves reveal deformation).

Algebra specifies the rule; geometry shows what the rule does.

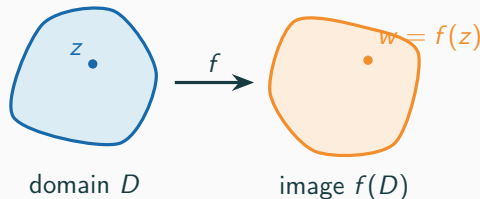


Complex Functions and Their Geometry

From points to transformations

Lecture 01 established the geometry and algebra of the complex plane. We now let one complex number depend on another:

$$f : D \subseteq \mathbb{C} \longrightarrow \mathbb{C}, \quad z = x + iy \longmapsto w = f(z).$$



A complex function transforms one plane into another.

Thus a complex function is simultaneously

- a rule assigning an output to each admissible input,
- a pair of real functions of two real variables,
- a transformation from the z -plane to the w -plane.

Domain, range, and singular points

The *domain* is the set of admissible inputs; the *range* is the set of attained outputs.

Examples

- $f(z) = z^2$: domain $\mathbb{C}$, range $\mathbb{C}$.
- $f(z) = 1/z$: domain $\mathbb{C} \setminus \{0\}$.
- A single-valued square root requires a chosen branch domain.

A formula alone does not tell the whole story: the domain is part of the definition of a complex function.

Real and imaginary components

Writing $w = u + iv$,

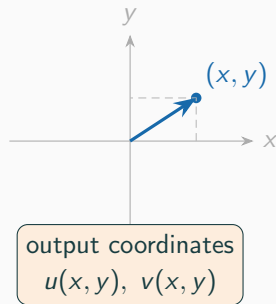
$$f(x + iy) = u(x, y) + iv(x, y),$$

where

$$u = \operatorname{Re} f(x + iy), \quad v = \operatorname{Im} f(x + iy).$$

Equivalently,

$$(x, y) \mapsto \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix}.$$



One complex rule is a coupled pair of real rules.

Key viewpoint

Questions about a complex function can often be translated into questions about the vector-valued real function $(u, v) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$.

Worked example: separating the components

For $f(z) = z^2 + 2z$, substitute $z = x + iy$:

$$\begin{aligned} f(z) &= (x + iy)^2 + 2(x + iy) \\ &= (x^2 - y^2 + 2x) + i(2xy + 2y). \end{aligned}$$

Therefore

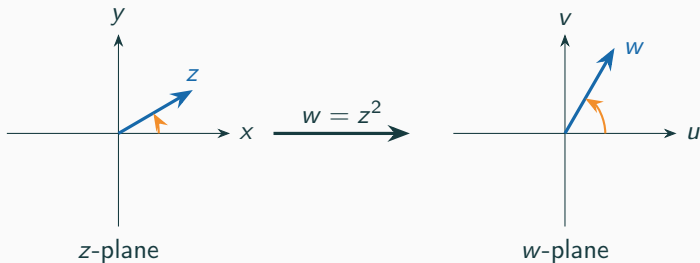
$$u(x, y) = x^2 - y^2 + 2x, \quad v(x, y) = 2xy + 2y.$$

This decomposition will later turn complex differentiability into conditions on the partial derivatives of u and v .

First view: map a point and its direction

For $w = z^2$, the argument doubles and the modulus is squared:

$$z = re^{i\theta} \implies w = r^2 e^{2i\theta}.$$

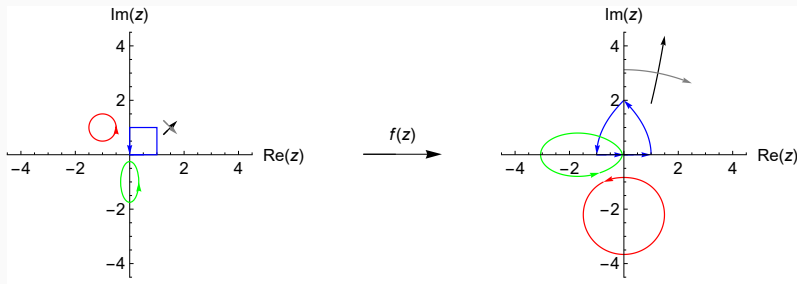


Next view: map complete geometric figures

A complex function acts pointwise on every curve or region. Under $w = z^2$:

- distances from the origin are squared,
- angles from the positive real axis are doubled.

Applying this rule to every point makes complete shapes stretch, rotate, or overlap as shown.



Effect of $f(z) = z^2$ on some geometric figures in the complex plane.

Visualization method I: Real and imaginary parts as two surfaces

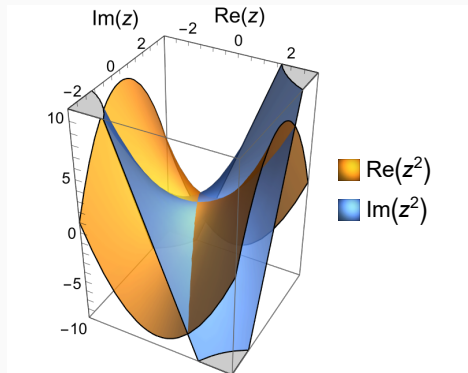
For $f(z) = z^2$,

$$u(x, y) = x^2 - y^2, \quad v(x, y) = 2xy.$$

The two surfaces encode different components of the same output:

- height on the orange surface gives $\operatorname{Re} f(z)$,
- height on the blue surface gives $\operatorname{Im} f(z)$.

This component view will later lead directly to the Cauchy–Riemann equations.



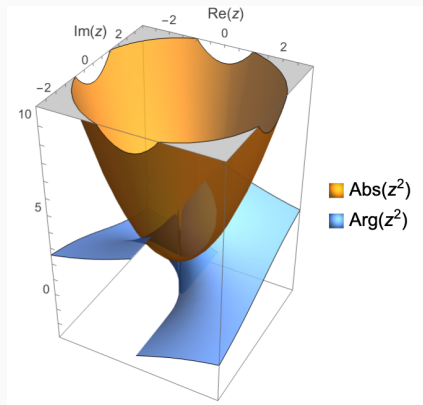
Real and imaginary parts of
 $w = f(z) = z^2$.

Visualization method II: Modulus and argument as two surfaces

Instead of real and imaginary parts, one can plot the modulus and argument separately:

- height on the orange surface gives $|z^2|$,
- height on the blue surface gives $\text{Arg}(z^2)$.

The apparent jump in the argument surface is only a branch-cut artifact; $f(z) = z^2$ is continuous everywhere.



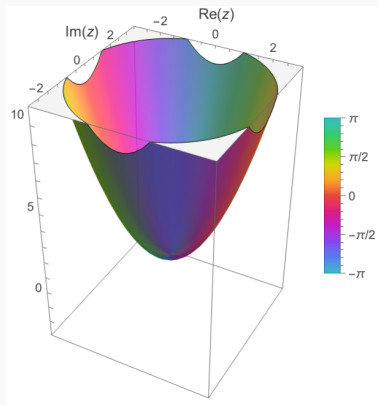
Absolute value and argument of
 $w = f(z) = z^2$.

Visualization method III: Three-dimensional view with color

`ComplexPlot3D` encodes both pieces of information in a single surface:

- height represents $|f(z)|$,
- color represents $\arg f(z)$.

This removes the confusion of two overlapping surfaces and makes poles and zeros easy to locate at a glance.



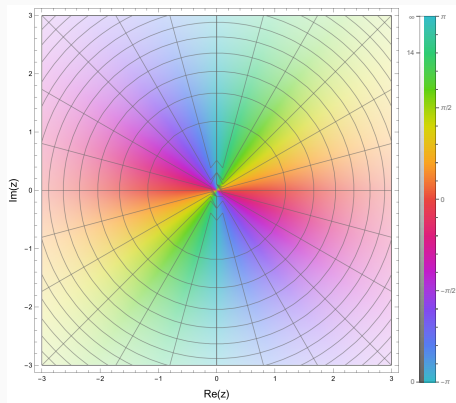
Absolute value and argument of
 $f(z) = z^2$ in one plot.

Visualization method IV: Two-dimensional view with color

The graph lives in four dimensions, so no single ordinary plot displays everything. Useful views include

- surfaces of u and v ,
- modulus and phase,
- domain coloring,
- images of grids and regions.

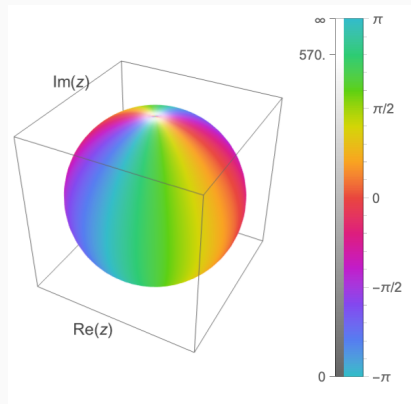
For z^2 , the color wheel turns twice around the origin.



Domain coloring of $f(z) = z^2$.

Visualization method V: The Riemann Sphere

Stereographic projection compactifies the plane onto the Riemann sphere. Plotting $f(z) = z^2$ on the sphere displays its global behavior, including the neighborhood of infinity.



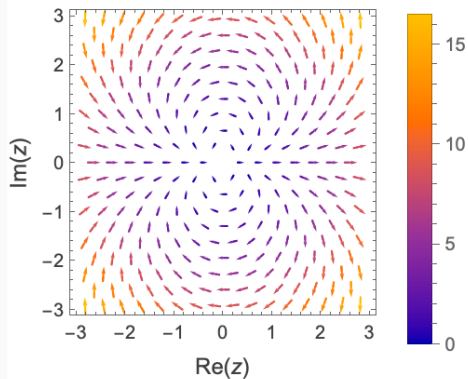
Plot of $f(z) = z^2$ projected onto the Riemann sphere.

Visualization method VI: Vector fields

A complex function can be drawn as a vector field:
at each point z attach the vector $(\operatorname{Re} f(z), \operatorname{Im} f(z))$.

- arrow length and color encode $|f(z)|$,
- arrow direction encodes $\arg f(z)$.

The resulting picture reveals the quadratic saddle structure of z^2 .

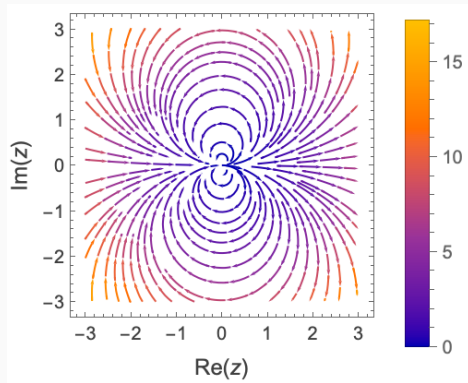


Vector plot of $f(z) = z^2$.

Visualization method VII: Streamlines

Joining the arrows of a vector field into continuous curves yields *streamlines*. They are still colored by modulus and illustrate the flow generated by the map.

Streamlines are especially useful for visualizing the orthogonal families of level curves that arise from analyticity.



Stream plot of $f(z) = z^2$.

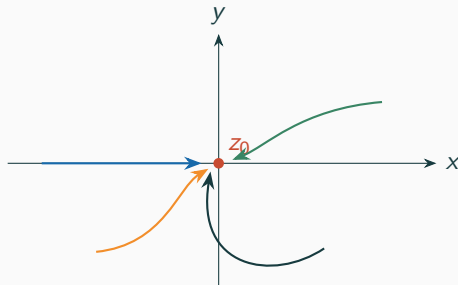
Limits and Continuity

Why complex limits are more demanding

On the real line, a point is approached from two directions. In the complex plane, it can be approached along infinitely many paths.



Real limit on the line.



Complex limit in the plane.

A complex limit exists only if every admissible path gives the same value.

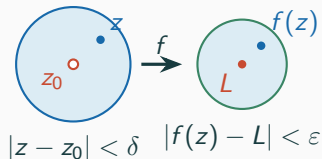
Definition of a complex limit

We write

$$\lim_{z \rightarrow z_0} f(z) = L$$

if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < |z - z_0| < \delta \implies |f(z) - L| < \varepsilon.$$



Geometric meaning

Every sufficiently small punctured disk around z_0 is mapped inside an arbitrarily small disk around L .

A complex limit is a two-variable limit

Let

$$f(z) = u(x, y) + iv(x, y), \quad z_0 = x_0 + iy_0, \quad L = a + ib.$$

Then

$$\lim_{z \rightarrow z_0} f(z) = L \iff \begin{cases} \lim_{(x,y) \rightarrow (x_0,y_0)} u(x, y) = a, \\ \lim_{(x,y) \rightarrow (x_0,y_0)} v(x, y) = b. \end{cases}$$

This criterion lets us use familiar tools from multivariable calculus.

Worked example: a limit that exists

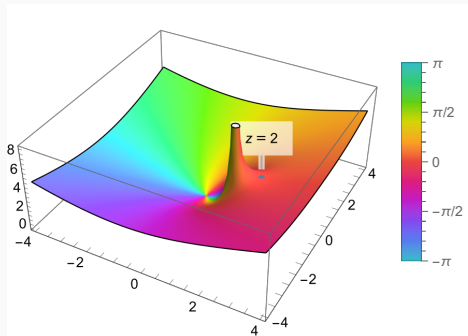
Evaluate

$$\lim_{z \rightarrow 2} \frac{z^2}{z-1}.$$

The marked point lies away from the singularity at $z = 1$. Since the denominator is nonzero at $z = 2$, the quotient is continuous there:

$$\lim_{z \rightarrow 2} \frac{z^2}{z-1} = \frac{4}{1} = 4.$$

The plot motivates the result; continuity proves it for every path.



The function $\frac{z^2}{z-1}$ and the point $z = 2$.

A practical test for non-existence

To disprove a proposed limit, it is enough to find two paths that give different limiting values.

Path test

If $z = \gamma_1(t) \rightarrow z_0$ gives L_1 and $z = \gamma_2(t) \rightarrow z_0$ gives $L_2 \neq L_1$, then

$$\lim_{z \rightarrow z_0} f(z) \text{ does not exist.}$$

Agreement along several paths is useful evidence, but it is not a proof that the full complex limit exists.

Worked example: path dependence

Consider

$$\lim_{z \rightarrow 0} \frac{z}{\bar{z}}.$$

Along the real axis, $z = x$:

$$\frac{z}{\bar{z}} = \frac{x}{x} = 1.$$

Along the imaginary axis, $z = iy$:

$$\frac{z}{\bar{z}} = \frac{iy}{-iy} = -1.$$

The two values disagree, so the limit does not exist.

Properties of limits

The properties of limits carry over directly from real analysis. If

$$\lim_{z \rightarrow z_0} f(z) = w_1 \quad \text{and} \quad \lim_{z \rightarrow z_0} g(z) = w_2,$$

then

$$\lim_{z \rightarrow z_0} (f(z) + g(z)) = w_1 + w_2,$$

$$\lim_{z \rightarrow z_0} (f(z) g(z)) = w_1 w_2,$$

provided the two limits on the right exist.

Moreover, if $\lim_{z \rightarrow z_0} f(z) = A$, then

$$\lim_{z \rightarrow z_0} \overline{f(z)} = \overline{A},$$

$$\lim_{z \rightarrow z_0} \operatorname{Re} f(z) = \operatorname{Re} A,$$

$$\lim_{z \rightarrow z_0} \operatorname{Im} f(z) = \operatorname{Im} A.$$

Worked example: real and imaginary parts of a limit

Evaluate

$$\lim_{z \rightarrow 1+i} (z^2 + \bar{z}).$$

Since z^2 and $\bar{z}$ are continuous at $z_0 = 1 + i$,

$$\lim_{z \rightarrow 1+i} z^2 = (1 + i)^2 = 2i,$$

$$\lim_{z \rightarrow 1+i} \bar{z} = \overline{(1 + i)} = 1 - i.$$

By the sum and conjugation rules,

$$\lim_{z \rightarrow 1+i} (z^2 + \bar{z}) = 2i + (1 - i) = 1 + i.$$

Consequently

$$\operatorname{Re}\left(\lim_{z \rightarrow 1+i} (z^2 + \bar{z})\right) = 1, \quad \operatorname{Im}\left(\lim_{z \rightarrow 1+i} (z^2 + \bar{z})\right) = 1.$$

A function f is continuous at z_0 if

$$\lim_{z \rightarrow z_0} f(z) = f(z_0).$$

Equivalently, when $f = u + iv$, both u and v are continuous at (x_0, y_0) .

- Sums and products of continuous functions are continuous.
- Quotients are continuous where the denominator is nonzero.
- Compositions of continuous functions are continuous wherever defined.

Worked example: removing a discontinuity

Define

$$f(z) = \frac{z^2 - 1}{z - 1}, \quad z \neq 1.$$

For $z \neq 1$, $f(z) = z + 1$, so

$$\lim_{z \rightarrow 1} f(z) = 2.$$

The original function has a removable discontinuity at $z = 1$. Defining

$$f(1) = 2$$

produces a function continuous on the whole complex plane.

Continuity depends on the chosen domain

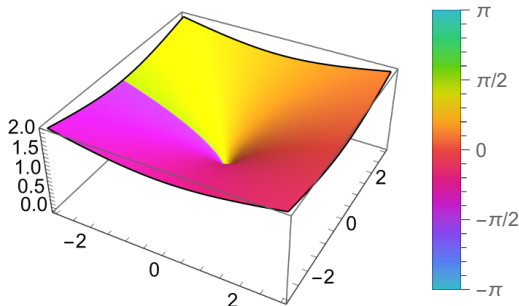
A multi-valued expression needs a branch before it becomes a function. For the principal square root, choose

$$\mathbb{C} \setminus (-\infty, 0].$$

The negative real axis is the branch cut. The plot keeps $|\sqrt{z}|$ as height and uses color for phase. The abrupt color change reveals the discontinuity across the cut.

Looking toward Lecture 03

The same domain issue governs $\text{Arg } z$, $\text{Log } z$, and complex powers.



The function $\sqrt{z}$ is not continuous along the negative real axis.

Properties of continuous functions

Algebraic operations

If $f(z)$ and $g(z)$ are continuous at a point a , then

- $f(z) + g(z)$ is continuous at a ;
- $f(z)g(z)$ is continuous at a ;
- $\frac{f(z)}{g(z)}$ is continuous at a , except when $g(a) = 0$.

Composition

If $g(z)$ is continuous at a and f is continuous at $g(a)$, then the composition $f(g(z))$ is continuous at a .

These rules let us assemble complicated continuous functions from simple pieces.

Worked example: continuity of a rational function

Let

$$f(z) = \frac{z^2 + 1}{z - i}.$$

- $z^2 + 1$ is a polynomial, hence continuous on the whole plane.
- $z - i$ is also a polynomial, continuous on $\mathbb{C}$.
- The denominator vanishes only at $z = i$.

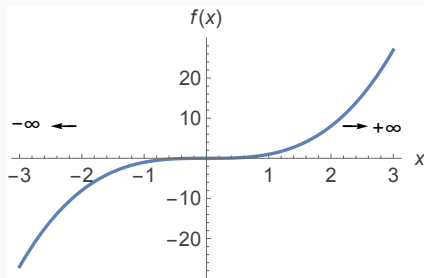
By the quotient rule, $R(z)$ is continuous on $\mathbb{C} \setminus \{i\}$.

Check the domain first

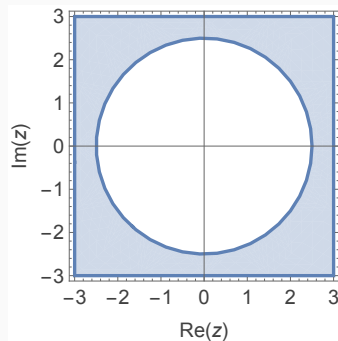
Algebraic rules do not remove a singularity or resolve a branch ambiguity.

The point at infinity

In real analysis, infinity can only be approached from two directions: $+\infty$ or $-\infty$.



In complex analysis, it can be approached from *any* directions.



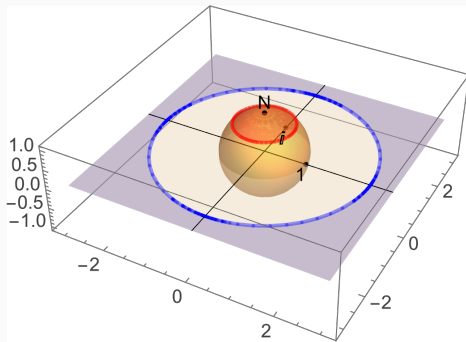
Neighborhoods of infinity

In order to treat infinity on the same footing as any other point, one needs to define a neighborhood of infinity.

A *neighborhood of infinity* is the set

$$\{z \in \mathbb{C} : |z| > R\}$$

for some $R > 0$. On the Riemann sphere this becomes a circular cap around the north pole.



A neighborhood of infinity on the Riemann sphere.

Definition

$$\lim_{z \rightarrow \infty} f(z) = A \iff \lim_{z \rightarrow 0} f\left(\frac{1}{z}\right) = A.$$

Similarly, a function tending to infinity means

$$\lim_{z \rightarrow z_0} f(z) = \infty \iff \lim_{z \rightarrow z_0} \frac{1}{f(z)} = 0,$$

$$\lim_{z \rightarrow \infty} f(z) = \infty \iff \lim_{z \rightarrow 0} \frac{1}{f(1/z)} = 0.$$

These definitions let us recycle familiar tools from finite limits.

Complex Differentiation

Definition of the complex derivative

Let f be defined in a neighborhood of z_0 . Its derivative is

$$f'(z_0) = \lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h},$$

provided that this complex limit exists.

The decisive difference

The increment $h \in \mathbb{C}$ may approach zero from any direction. Every direction must produce the same derivative.

Geometrically, a derivative measures the local linear approximation:

$$f(z_0 + h) \approx f(z_0) + f'(z_0)h.$$

So the map looks like a rotation by $\arg f'(z_0)$ and a scaling by $|f'(z_0)|$, exactly the kind of geometry emphasized in Lecture 01.

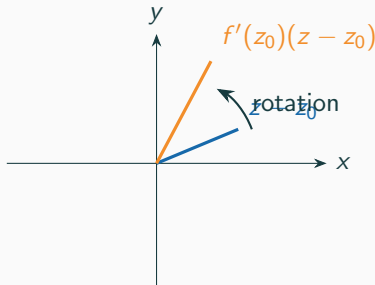
Local geometry: magnification and rotation

If $f'(z_0) \neq 0$, then near z_0 the map is approximately

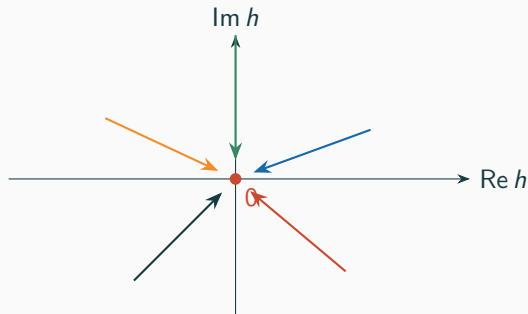
$$z \mapsto f(z_0) + f'(z_0)(z - z_0).$$

This is the complex version of a linear map in the plane:

- $|f'(z_0)|$: local scale factor,
- $\arg f'(z_0)$: local rotation angle,
- the derivative captures the infinitesimal geometry of the map.



One difference quotient, infinitely many directions



Real differentiability controls one line of approach. Complex differentiability requires compatibility across the entire plane.

Worked example: $f(z) = z^2$

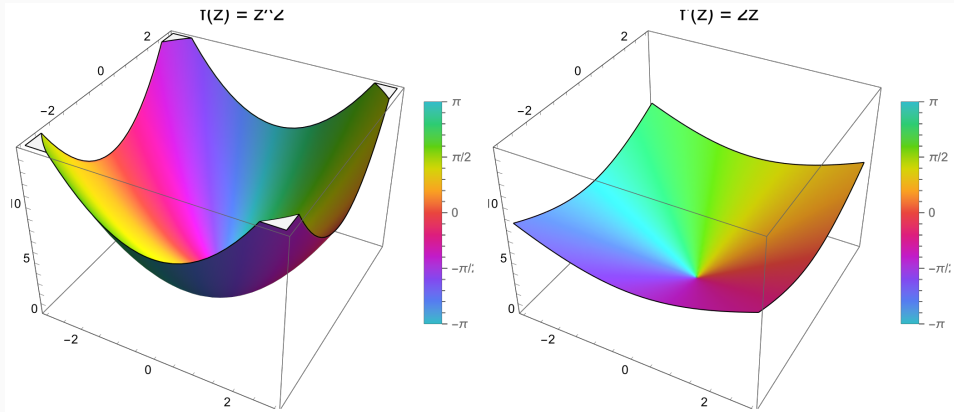
For any $z_0 \in \mathbb{C}$,

$$\begin{aligned} f'(z_0) &= \lim_{h \rightarrow 0} \frac{(z_0 + h)^2 - z_0^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{2z_0h + h^2}{h} \\ &= \lim_{h \rightarrow 0} (2z_0 + h) = 2z_0. \end{aligned}$$

The simplified expression has the same limit along every path, so

$$\frac{d}{dz} z^2 = 2z \quad \text{for all } z \in \mathbb{C}.$$

The function and its derivative



In these complex surfaces, height represents modulus and color represents argument. The derivative $2z$ records the local change produced by z^2 .

Counterexample: $f(z) = \bar{z}$

At z_0 , the difference quotient is

$$\frac{\overline{z_0 + h} - \bar{z}_0}{h} = \frac{\bar{h}}{h}.$$

If h is real, the quotient equals 1. If h is purely imaginary, it equals -1 .

$f'(z_0)$ does not exist.

Conjugation is continuous everywhere but complex differentiable nowhere.

Differentiability implies continuity

If $f'(z_0)$ exists, then

$$f(z) - f(z_0) = (z - z_0) \frac{f(z) - f(z_0)}{z - z_0}.$$

As $z \rightarrow z_0$, the first factor tends to zero and the second tends to $f'(z_0)$. Hence

$$\lim_{z \rightarrow z_0} (f(z) - f(z_0)) = 0.$$

Complex differentiability at z_0 implies continuity at z_0 , but continuity does not imply complex differentiability.

Differentiation rules

Whenever the relevant derivatives exist,

$$\frac{d}{dz}(f(z) + g(z)) = f'(z) + g'(z),$$

$$\frac{d}{dz}(f(z)g(z)) = f'(z)g(z) + f(z)g'(z),$$

$$\frac{d}{dz}\left(\frac{f}{g}\right) = \frac{f'(z)g(z) - f(z)g'(z)}{g^2(z)}, \quad g \neq 0,$$

$$\frac{d}{dz}f(g(z)) = f'(g(z))g'(z).$$

The algebra is familiar; the difficult question is whether the complex derivatives exist on the domain under consideration.

Worked example: a rational function

Let

$$f(z) = \frac{z^2 + 1}{z - i}, \quad z \neq i.$$

The quotient rule gives

$$\begin{aligned} f'(z) &= \frac{2z(z - i) - (z^2 + 1)}{(z - i)^2} \\ &= \frac{(z - i)^2}{(z - i)^2} = 1. \end{aligned}$$

The derivative exists throughout $\mathbb{C} \setminus \{i\}$. The simplified formula does not automatically add the excluded point back to the domain.

Differentiable, analytic, and entire

Differentiable at z_0 : $f'(z_0)$ exists.

Analytic near z_0 : f is differentiable throughout some open neighborhood of z_0 .

Analytic on D : f is differentiable at every point of the open set D .

Entire: f is analytic on all of $\mathbb{C}$.

The term *holomorphic* is synonymous with *analytic* in this course.

Analyticity is far more restrictive than ordinary real differentiability. It leads to

- strong relations between the real and imaginary parts,
- local angle preservation away from critical points,
- derivatives of every order and local power-series representations,
- powerful integral theorems studied in later lectures.

The first of these consequences can already be derived from the definition.

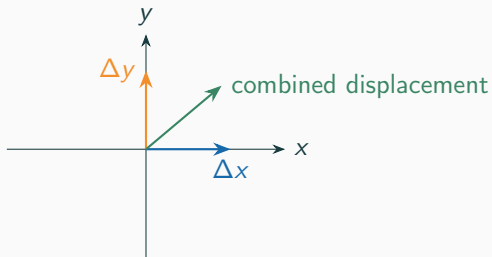
Analyticity and the Cauchy–Riemann Equations

Same derivative from different direction

Suppose

$$f(z) = f(x + iy) = u(x, y) + iv(x, y)$$

is analytic. This means that its derivative $f'(z)$ exists. Therefore, it must be the same whether it is computed as a limit from every direction.



same derivative from every direction

Derivation along the real direction

Choose a real increment $h = \Delta x$ with $\Delta x \in \mathbb{R}$, $\Delta x \rightarrow 0$:

$$\begin{aligned}f'(z) &= \lim_{\Delta x \rightarrow 0} \frac{f(z + \Delta x) - f(z)}{\Delta x} \\&= \lim_{\Delta x \rightarrow 0} \frac{u(x + \Delta x, y) + iv(x + \Delta x, y) - u(x, y) - iv(x, y)}{\Delta x} \\&= \lim_{\Delta x \rightarrow 0} \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} + i \lim_{\Delta x \rightarrow 0} \frac{v(x + \Delta x, y) - v(x, y)}{\Delta x} \\&= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}.\end{aligned}$$

The real-direction derivative picks out the horizontal change in the plane, so this is the geometry of a map in the x -direction.

Derivation along the imaginary direction

Now choose a imaginary increment $h = i\Delta y$ with $\Delta y \in \mathbb{R}$, $\Delta y \rightarrow 0$:

$$\begin{aligned}f'(z) &= \lim_{\Delta y \rightarrow 0} \frac{f(z + i\Delta y) - f(z)}{i\Delta y} \\&= \lim_{\Delta y \rightarrow 0} \frac{u(x, y + \Delta y) + iv(x, y + \Delta y) - u(x, y) - iv(x, y)}{i\Delta y} \\&= \lim_{\Delta y \rightarrow 0} \frac{u(x, y + \Delta y) - u(x, y)}{i\Delta y} + \lim_{\Delta y \rightarrow 0} \frac{iv(x, y + \Delta y) - iv(x, y)}{i\Delta y} \\&= -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.\end{aligned}$$

The imaginary-direction derivative picks out the Vertical change in the plane, so this is the geometry of a map in the y -direction.

The Cauchy–Riemann equations

Since both limits represent the same $f'(z)$,

$$\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

Hence the Cauchy–Riemann equations follow by matching real and imaginary parts:

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.}$$

When these hold, the derivative can be written in four equivalent ways:

$$f'(z) = u_x + i v_x = v_y - i u_y = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}.$$

Quick check

For $f(z) = z^2$ we obtain $f'(z) = 2x + i(2y) = 2z$, exactly as expected.

When are the equations sufficient?

Necessary condition

If either Cauchy–Riemann equation fails at a point, f is not complex differentiable there.

Course-level sufficiency theorem

Suppose u_x, u_y, v_x, v_y exist and are continuous in a neighborhood of (x_0, y_0) . If the Cauchy–Riemann equations hold at (x_0, y_0) , then $f = u + iv$ is differentiable at $z_0 = x_0 + iy_0$.

Thus, for functions with continuous first partial derivatives on an open region,

Cauchy–Riemann throughout the region $\iff$ analyticity on the region.

To determine where $f(z)$ is analytic:

1. Write $f(x + iy) = u(x, y) + iv(x, y)$.
2. Compute u_x, u_y, v_x, v_y .
3. Solve $u_x = v_y$ and $u_y = -v_x$.
4. Check continuity of the first partial derivatives.
5. Distinguish isolated differentiability from analyticity on an open set.

This procedure is especially useful when a formula contains $\bar{z}$, $|z|$, x , or y .

Worked example: verify analyticity of z^3

Expand

$$f(z) = (x + iy)^3 = (x^3 - 3xy^2) + i(3x^2y - y^3).$$

Hence

$$u = x^3 - 3xy^2, \quad v = 3x^2y - y^3.$$

The partial derivatives are

$$u_x = 3x^2 - 3y^2 = v_y, \quad u_y = -6xy = -v_x.$$

They are continuous everywhere, so z^3 is entire.

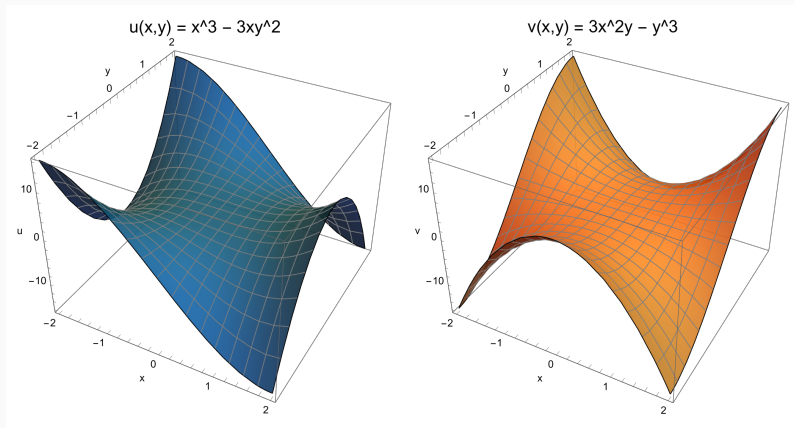
Recover the derivative from u and v

Using $f'(z) = u_x + iv_x$,

$$\begin{aligned} f'(z) &= (3x^2 - 3y^2) + i(6xy) \\ &= 3(x^2 - y^2 + 2ixy) \\ &= 3(x + iy)^2 = 3z^2. \end{aligned}$$

The result agrees with the ordinary power rule, now justified by the Cauchy–Riemann equations.

The two component surfaces of z^3



The surfaces are different, but their slopes are coupled everywhere by the Cauchy–Riemann equations.

Worked example: detect non-analyticity

For $f(z) = \bar{z} = x - iy$,

$$u = x, \quad v = -y.$$

Therefore

$$u_x = 1, \quad v_y = -1.$$

The first Cauchy–Riemann equation fails at every point. Hence $\bar{z}$ is nowhere complex differentiable, in agreement with the path calculation.

A subtle point: differentiable does not mean analytic

Let $f(z) = |z|^2 = x^2 + y^2$, so $u = x^2 + y^2$ and $v = 0$. The Cauchy–Riemann equations hold only at $(0, 0)$, and

$$f'(0) = \lim_{h \rightarrow 0} \frac{|h|^2}{h} = \lim_{h \rightarrow 0} \bar{h} = 0.$$

Thus f is differentiable at 0, but it is not analytic at 0: there is no neighborhood of 0 on which it is differentiable everywhere.

Harmonic Consequences

Analytic components are harmonic

Assume the required second partial derivatives are continuous. Differentiating the Cauchy–Riemann equations gives

$$u_{xx} = v_{yx}, \quad u_{yy} = -v_{xy}.$$

Therefore

$$u_{xx} + u_{yy} = 0.$$

Similarly,

$$v_{xx} + v_{yy} = 0.$$

Functions satisfying Laplace's equation are called *harmonic*.

Worked example: harmonic components of z^2

For

$$f(z) = z^2 = (x^2 - y^2) + i(2xy),$$

we have

$$u = x^2 - y^2, \quad v = 2xy.$$

Then

$$u_{xx} + u_{yy} = 2 - 2 = 0, \quad v_{xx} + v_{yy} = 0 + 0 = 0.$$

Both components are harmonic, as required by analyticity.

Level curves reveal the Cauchy–Riemann structure

For $f(z) = z^2$,

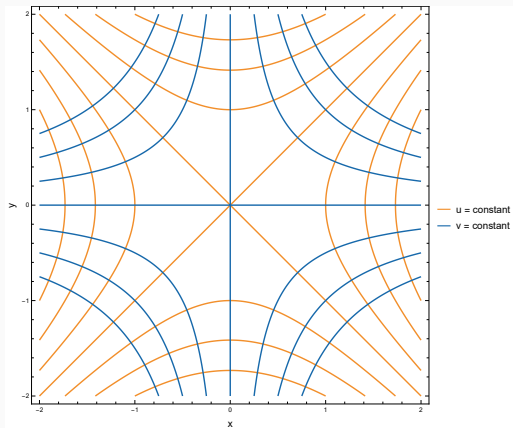
$$\nabla u = (2x, -2y),$$

$$\nabla v = (2y, 2x).$$

Hence

$$\nabla u \cdot \nabla v = 0.$$

The level curves of u and v meet at right angles wherever the gradients are nonzero.

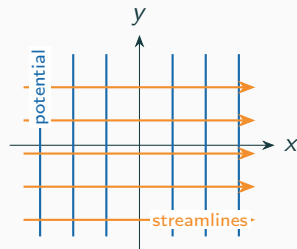


Why engineers care about harmonic functions

Laplace's equation models steady-state fields in many applications:

- electrostatic potential in charge-free regions,
- steady temperature in a homogeneous plate,
- incompressible and irrotational planar flow.

Complex analysis packages two coupled harmonic fields into one analytic function.



Orthogonal field families

This connection will reappear in conformal mapping and engineering applications.

Summary: What we have learned

- A complex function is a map $f : D \subseteq \mathbb{C} \rightarrow \mathbb{C}$, equivalently a pair (u, v) of real functions.
- A complex limit must be independent of every path of approach; continuity means $\lim_{z \rightarrow z_0} f(z) = f(z_0)$.
- A complex derivative is a direction-independent difference quotient.
- Analyticity means differentiability throughout an open neighborhood.
- The Cauchy–Riemann equations connect analyticity to the partial derivatives of u and v .
- The real and imaginary parts of an analytic function are harmonic.

A compact decision chain

For a candidate function $f = u + iv$: identify the domain $\implies$ check continuity and singular points $\implies$ test the Cauchy–Riemann equations $\implies$ verify the regularity assumptions $\implies$ state the region of analyticity and compute $f'(z)$.

The next lecture studies the principal elementary complex functions:

- the complex exponential and trigonometric functions,
- roots and the argument function,
- logarithms and complex powers.

The tools developed here will determine their continuity, derivatives, analytic domains, and branch cuts.

Next » Lecture 03 — Elementary Complex Functions