

L03 — Elementary Complex Functions

Engineering Mathematics I

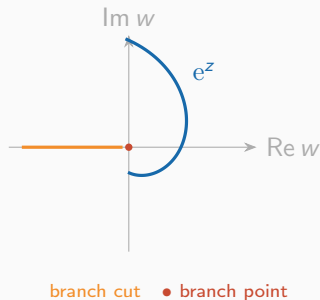
Prof. Tianyu DONG & Jiawei WANG

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School of Electrical Engineering, Xi'an Jiaotong University

Road map

- From analyticity to elementary functions
- The argument functions
- The complex exponential, trigonometric and hyperbolic functions
- The complex logarithm, complex powers and branch choices
- Roots and rational powers
- Inverse trigonometric and hyperbolic functions
- A unified view



From Analyticity to Elementary Functions

From a test to a toolbox

In the previous lecture (L02), analyticity was a property to be tested. We now build a library of functions whose behavior can be used directly.



For every function we ask:

- What is its value?
- What geometry does it create?
- What is its derivative?
- Where is it analytic?
- Does it require a branch choice?

Single-valued and multi-valued behavior

Single-valued functions

Each input has exactly one output:

$$e^z, \quad \sin z, \quad \cos z, \quad \sinh z, \quad \cosh z.$$

These functions are analytic on all of $\mathbb{C}$.

Naturally multi-valued objects

A point may have several possible values:

$$z^{1/n}, \quad \arg z, \quad \log z, \quad z^\alpha.$$

A *branch* selects one value on a suitable domain.

Branches are not software conventions; they are required by the geometry of the complex plane.

The Argument Functions

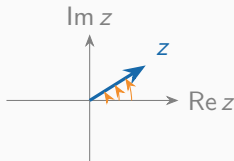
The argument of a complex number is naturally multi-valued

We know that every $z \neq 0$ factors as

$$z = |z|(\cos \theta + i \sin \theta) = |z| e^{i\theta} = |z| e^{i(\theta + 2k\pi)}, \quad k \in \mathbb{Z}.$$

The angle θ is *an* argument of z , and $\theta, \theta + 2\pi, \theta + 4\pi, \dots$ all describe the same z . The lowercase symbol $\arg z$ denotes the complete family of angles:

$$\arg z = \theta + 2k\pi, \quad k \in \mathbb{Z}.$$



Possible angles: $\theta + 2k\pi, k \in \mathbb{Z}$.

A function $\text{Arg } z$ must pick a single value — a *branch* — otherwise it is multi-valued.

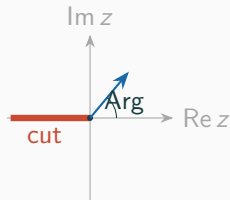
The principal branch

The common convention (e.g., this course, Wolfram Language) is

$$-\pi < \text{Arg } z \leq \pi.$$

Then θ is unique, and the branch cut of Arg lies along the negative real axis. Arg is undefined at 0 (the branch point).

Coloring the plane by $\text{Arg } z$ shows a jump of 2π across $(-\infty, 0]$.



See the Wolfram notebook for details.

The jump in the principal argument

Approaching a negative real number from above,

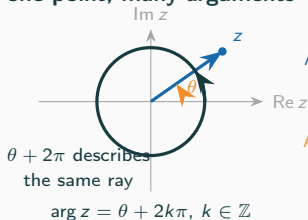
$$\operatorname{Arg} z \rightarrow \pi.$$

Approaching from below,

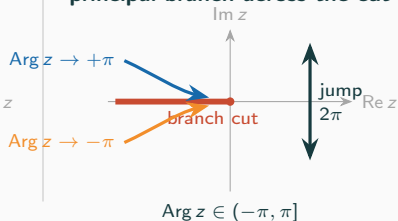
$$\operatorname{Arg} z \rightarrow -\pi.$$

The jump is 2π .

one point, many arguments



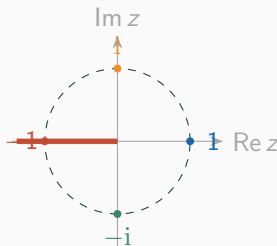
principal branch across the cut



Special values of Arg

$$\operatorname{Arg}(1) = 0, \quad \operatorname{Arg}(i) = \frac{\pi}{2}, \quad \operatorname{Arg}(-1) = \pi, \quad \operatorname{Arg}(-i) = -\frac{\pi}{2}.$$

Note the asymmetry: $\operatorname{Arg}(-1) = \pi$, not $-\pi$, because the interval is $(-\pi, \pi]$.



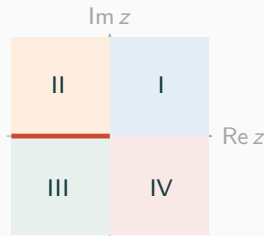
Important distinction

A principal value can be assigned on $\mathbb{C} \setminus \{0\}$, but it is discontinuous across the negative real axis. Removing that ray gives a continuous branch.

Computing $\text{Arg } z$ — use atan2

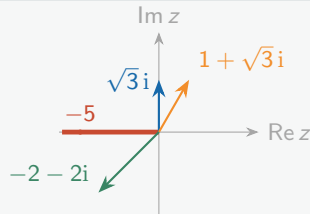
For $z = x + iy \neq 0$, $\text{Arg } z = \text{atan2}(y, x)$, never $\arctan(y/x)$ alone — the latter loses the quadrant.

- $x > 0$: $\text{Arg} = \arctan(y/x) \in (-\pi/2, \pi/2)$
- $x < 0, y \geq 0$: $\text{Arg} = \arctan(y/x) + \pi$
- $x < 0, y < 0$: $\text{Arg} = \arctan(y/x) - \pi$
- $x = 0, y > 0$: $\text{Arg} = \pi/2$
- $x = 0, y < 0$: $\text{Arg} = -\pi/2$



Computing $\text{Arg } z$

- $\sqrt{3}i \Rightarrow \text{Arg} = \pi/2$
- $1 + \sqrt{3}i \Rightarrow \text{Arg} = \pi/3$
- $-2 - 2i \Rightarrow \text{Arg} = -3\pi/4$
- $-5 \Rightarrow \text{Arg} = \pi$



Algebra of Arg

Up to integer multiples of 2π ,

$$\arg(zw) = \arg z + \arg w, \quad \arg(1/z) = -\arg z.$$

For the *principal* value these hold only after reducing into $(-\pi, \pi]$:

$$\text{Arg}(zw) = \text{Arg } z + \text{Arg } w - 2\pi n, \quad n \in \{-1, 0, 1\}.$$

Equality may fail before reduction. Crossing the cut is why Arg is discontinuous. **Argument rules require reduction modulo 2π .**

Example

- Let $z_1 = z_2 = e^{3\pi i/4}$. Then $\text{Arg } z_1 + \text{Arg } z_2 = \frac{3\pi}{2}$, $\text{Arg}(z_1 z_2) = \text{Arg}(-i) = -\frac{\pi}{2}$. The answers differ by 2π .
- A typical failure: $\text{Arg}(-1) + \text{Arg}(-1) = \pi + \pi = 2\pi$, but $\text{Arg}((-1)(-1)) = \text{Arg}(1) = 0$. The identity is restored after subtracting 2π .

Real differentiability of $\text{Arg } z$

On the cut plane $\mathbb{C} \setminus (-\infty, 0]$, write $z = x + iy$ and

$$u(x, y) = \text{Arg}(x + iy) = \text{atan2}(y, x).$$

Since $x^2 + y^2 > 0$, u is a smooth real-valued function there, with

$$u_x = -\frac{y}{x^2 + y^2}, \quad u_y = \frac{x}{x^2 + y^2}.$$

Thus $\text{Arg } z$ is differentiable as a function of the two real variables (x, y) at every point of its branch domain. Its differential is

$$d(\text{Arg } z) = -\frac{y}{|z|^2} dx + \frac{x}{|z|^2} dy.$$

But real differentiability is not complex differentiability

A complex derivative requires the Cauchy–Riemann equations as well.

Is $\text{Arg } z$ complex differentiable?

Regard the real-valued function as a complex-valued function

$$f(z) = \text{Arg } z = u(x, y) + iv(x, y), \quad v(x, y) = 0.$$

The Cauchy–Riemann equations would require

$$u_x = v_y = 0, \quad u_y = -v_x = 0.$$

However, on the cut plane,

$$(u_x, u_y) = \frac{1}{x^2 + y^2}(-y, x) \neq (0, 0),$$

because $z \neq 0$. Therefore the equations fail at every point.

Conclusion

$\text{Arg } z$ is real-differentiable on $\mathbb{C} \setminus (-\infty, 0]$, but it is nowhere complex differentiable there. It is also discontinuous across its branch cut and undefined at the branch point 0.

Summary: the function $\text{Arg } z$

Question	Answer
What is its value?	For $z = re^{i\theta} \neq 0$, the principal value is $\text{Arg } z = \theta \in (-\pi, \pi]$; equivalently, $\text{Arg}(x + iy) = \text{atan2}(y, x)$.
What geometry does it create?	Its level curves are rays from the origin. Along a ray the value is constant; moving once counterclockwise around 0 increases the continuously unwrapped angle by 2π .
What is its derivative?	As a real function, $d(\text{Arg } z) = (-y dx + x dy)/ z ^2$. It has no complex derivative anywhere on its branch domain.
Where is it analytic?	Nowhere: $\text{Arg } z$ is not holomorphic on any open subset of its domain, although it is smooth as a real-valued function on the cut plane.
Does it require a branch choice?	Yes. The multi-valued $\arg z$ contains $\theta + 2\pi k$; the principal branch chooses $(-\pi, \pi]$ and cuts along $(-\infty, 0]$.

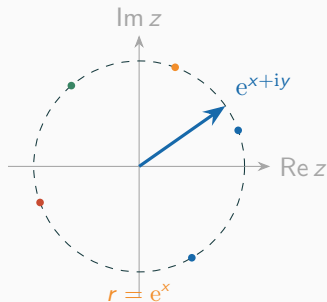
Main message

$\text{Arg } z$ measures direction, not a holomorphic change of complex coordinates: it is smooth away from the cut but nowhere analytic.

The Complex Exponential

Key points

- Real versus complex elementary functions
- Definitions of e^z
- Algebraic properties and $2\pi i$ -periodicity
- Geometry of $z \mapsto e^z$



From real to complex

You already know $\exp$, $\sin$, and $\cos$ on the real line.

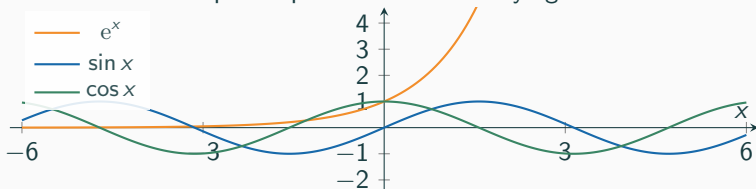
On $\mathbb{R}$

- $\sin x$, $\cos x$ are *bounded* and *periodic*
- $\exp x$ is *unbounded* and *not* periodic
- the three functions look unrelated

On $\mathbb{C}$

- they become three faces of one object
- $\sin z$ and $\cos z$ are *unbounded*
- $\exp z$ becomes *periodic*

The complex exponential is the unifying device.



Three equivalent definitions

A function $f : \mathbb{C} \rightarrow \mathbb{C}$ is the complex exponential if it satisfies any one of:

1. **ODE characterisation.**

$$f' = f, \quad f(0) = 1.$$

2. **Limit characterisation.**

$$f(z) = \lim_{n \rightarrow \infty} \left(1 + \frac{z}{n}\right)^n.$$

3. **Power series** (the working definition).

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots.$$

The radius of convergence is ∞ : the series defines an entire function.

From the series to Cartesian form

Write $z = x + iy$ with $x, y \in \mathbb{R}$. Separating even/odd powers, or using uniqueness of the ODE, yields the fundamental identity

$$e^{x+iy} = e^x (\cos y + i \sin y).$$

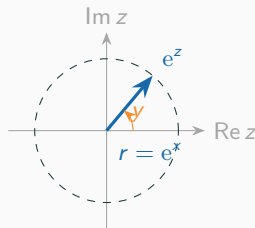
Consequently

$$\operatorname{Re}(e^z) = e^x \cos y,$$

$$\operatorname{Im}(e^z) = e^x \sin y,$$

$$|e^z| = e^x = e^{\operatorname{Re} z},$$

$$\arg(e^z) = y = \operatorname{Im} z \pmod{2\pi}.$$



Rule of thumb: the real part of the exponent sets the *size*;
the imaginary part sets the *angle*.

Worked example: compute $e^{2\pm 3i}$ in Cartesian form

- $e^{2+3i} = e^2(\cos 3 + i \sin 3) \approx -7.315 + 1.043i$;
- $e^{2-3i} \approx -7.315 - 1.043i$ (same modulus e^2 , opposite imaginary part).

The addition formula

The identity familiar from real analysis remains valid for all $z, w \in \mathbb{C}$:

$$e^{z+w} = e^z e^w.$$

Proof sketch. Both sides, as functions of z with w fixed, solve $f' = f$ with $f(0) = e^w$. Uniqueness gives equality.

Immediate consequences:

- $e^0 = 1$, and $e^{-z} = 1/e^z$
- $e^z \neq 0$ for every $z \in \mathbb{C}$ (if $e^{z_0} = 0$ then $1 = e^{z_0} e^{-z_0} = 0$)
- $\overline{e^z} = e^{\bar{z}}$
- $e^{nz} = (e^z)^n$ for $n \in \mathbb{Z}$

Periodicity: the new phenomenon

Because $\cos$ and $\sin$ have period 2π ,

$$e^{z+2\pi i} = e^z e^{2\pi i} = e^z (\cos 2\pi + i \sin 2\pi) = e^z.$$

Hence e^z is periodic with imaginary period $2\pi i$:

$$e^{z+2\pi in} = e^z \quad \text{for all } n \in \mathbb{Z}.$$

Kernel of the exponential

$e^z = 1$ if and only if $z = 2\pi in$ for some $n \in \mathbb{Z}$. More generally, $e^z = e^w$ iff $z - w = 2\pi in$.

The complex exponential is therefore *not* injective: it repeats on every horizontal strip of height 2π .

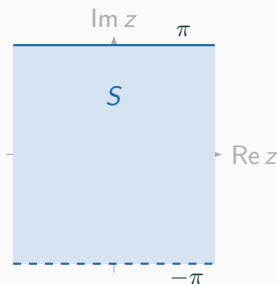
The fundamental strip

Every nonzero $w \in \mathbb{C}$ is e^z for infinitely many z . A convenient complete set of representatives is the strip

$$S = \{z : -\pi < \operatorname{Im} z \leq \pi\}.$$

On S the exponential is bijective onto $\mathbb{C} \setminus \{0\}$.

This strip is exactly the range of the principal logarithm (next lecture).

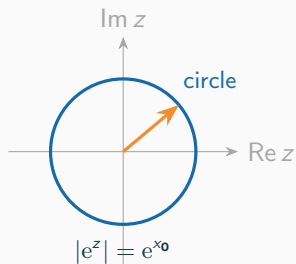


What $z \mapsto e^z$ does to lines

Write $z = x + iy$. Then e^z has modulus e^x and argument y .

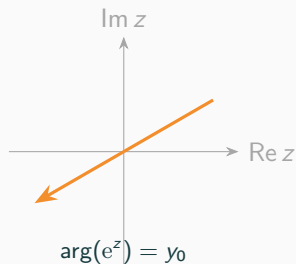
Vertical line

$x = x_0$ (fixed real part)



Horizontal line

$y = y_0$ (fixed imag. part)



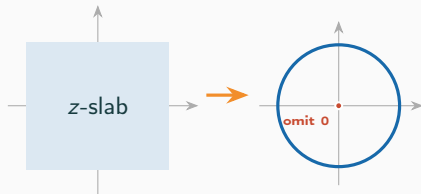
Rectangles in the z -plane become annular sectors in the w -plane. Horizontal width $\mapsto$ radial stretch; vertical height $\mapsto$ rotation.

Visualization of the complex exponential

See the Wolfram notebook for details.

The exponential as a covering map

- e^z maps $\mathbb{C}$ onto $\mathbb{C} \setminus \{0\}$
- it is locally conformal ($(e^z)' = e^z \neq 0$)
- it wraps each slab $\{z : \alpha < \operatorname{Im} z \leq \alpha + 2\pi\}$ once onto $\mathbb{C} \setminus \{0\}$
- 0 is an omitted value (Picard's small theorem in miniature)



Differentiability of e^z

Write $z = x + iy$. From the Cartesian form,

$$e^z = u(x, y) + iv(x, y), \quad u = e^x \cos y, \quad v = e^x \sin y.$$

The partial derivatives are

$$u_x = e^x \cos y = v_y, \quad u_y = -e^x \sin y = -v_x.$$

Hence the Cauchy–Riemann equations hold everywhere in $\mathbb{C}$, and the partial derivatives are continuous. Therefore e^z is complex differentiable everywhere, with

$$\frac{d}{dz} e^z = u_x + iv_x = e^x (\cos y + i \sin y) = e^z.$$

Equivalently,

$$\lim_{h \rightarrow 0} \frac{e^{z+h} - e^z}{h} = e^z \lim_{h \rightarrow 0} \frac{e^h - 1}{h} = e^z.$$

Engineering interpretation

$e^{(\sigma+i\omega)t} = e^{\sigma t}(\cos \omega t + i \sin \omega t)$ unifies exponential growth/decay with oscillation.

Summary: the function e^z

Question	Answer
What is its value?	For $z = x + iy$, $e^z = e^x(\cos y + i \sin y)$; hence $ e^z = e^x$ and $\arg(e^z) = y \pmod{2\pi}$.
What geometry does it create?	Vertical lines $x = x_0$ map to circles $ w = e^{x_0}$; horizontal lines $y = y_0$ map to rays of argument y_0 . Rectangles map to annular sectors.
What is its derivative?	$\frac{d}{dz}e^z = e^z$, which never vanishes.
Where is it analytic?	Everywhere on $\mathbb{C}$: e^z is entire and locally conformal.
Does it require a branch choice?	No. The exponential is single-valued. It is periodic rather than branched: $e^{z+2\pi in} = e^z$ for $n \in \mathbb{Z}$.

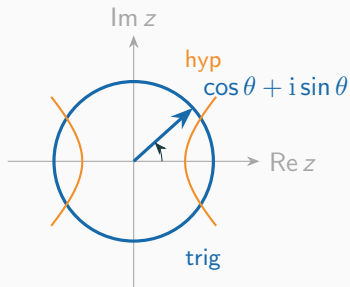
Main message

e^z is a globally analytic map that turns translation in the real direction into scaling and translation in the imaginary direction into rotation.

Trigonometric and Hyperbolic Functions

Key points

- Complex sine and cosine
- Euler's formula and identities
- Hyperbolic functions



Power series definitions

Copy the real Taylor series, now for $z \in \mathbb{C}$:

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, \quad \sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}.$$

Both series converge everywhere, so $\sin$ and $\cos$ are entire.

Differentiating termwise recovers the familiar calculus:

$$(\sin z)' = \cos z, \quad (\cos z)' = -\sin z, \quad \sin 0 = 0, \quad \cos 0 = 1.$$

Sine and cosine in the complex plane: Exponential expressions

Comparing the three series (or solving the ODE) gives Euler's inversion

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

These formulae are now *definitions* as much as identities: they extend sin and cos off the real axis.

In particular, for real θ ,

$$e^{i\theta} = \cos \theta + i \sin \theta,$$

which is Euler's formula.

Properties inherited from e^z :

- Both $\sin z$ and $\cos z$ are entire
- Pythagorean identity: $\sin^2 z + \cos^2 z = 1$
- Addition formulas: $\sin(z + w) = \sin z \cos w + \cos z \sin w$, $\sin(-z) = -\sin z$, $\cos(-z) = \cos z$;
 $\overline{\sin z} = \sin \bar{z}$, $\overline{\cos z} = \cos \bar{z}$ etc.

Cartesian splitting

Let $z = x + iy$. The addition formulae (justified later) give

$$\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y,$$

$$\cos(x + iy) = \cos x \cosh y - i \sin x \sinh y.$$

Consequently,

- On the real axis ($y = 0$) one recovers $\sin x$, $\cos x$
- On the imaginary axis, $\sin(iy) = i \sinh y$, $\cos(iy) = \cosh y$
- $|\sin z|$ and $|\cos z|$ grow like $\frac{1}{2}e^{|y|}$ as $|y| \rightarrow \infty$

Key difference: $|\sin z|$, $|\cos z|$ grow exponentially away from the real axis. Sine and cosine are unbounded on $\mathbb{C}$.

Worked example: sin and cos of a complex number

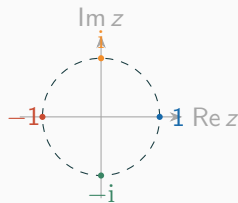
Compute $\cos(2 + i)$: $\cos(2 + i) = \cos 2 \cosh 1 - i \sin 2 \sinh 1 \approx -0.6421 - 1.0686 i$.

Likewise, $\sin\left(\frac{\sqrt{2}}{2}(1 + i)\right) = \sin \frac{\sqrt{2}}{2} \cosh \frac{\sqrt{2}}{2} + i \cos \frac{\sqrt{2}}{2} \sinh \frac{\sqrt{2}}{2}$.

Special values of the exponential

$$e^0 = 1, \quad e^{i\pi/2} = i, \quad e^{i\pi} = -1, \quad e^{3i\pi/2} = -i, \\ e^{2\pi i} = 1, \quad e^{i\pi} + 1 = 0.$$

The last identity is Euler's jewel: five fundamental constants in one equation.



Worked examples

- Solve $e^{i\theta} = i$: Write $i = e^{i\pi/2}$, then $e^{i\theta} = e^{i\pi/2}$. $\iff \theta = \frac{\pi}{2} + 2\pi n, \quad n \in \mathbb{Z}$. The principal value is $\theta = \pi/2$.
- Likewise, $e^{i\theta} = \frac{\sqrt{3}}{2} + \frac{i}{2} = e^{i\pi/6}$. $\iff \theta = \frac{\pi}{6} + 2\pi n, \quad n \in \mathbb{Z}$.

Euler's formula

For every complex z (in particular every real θ),

$$e^{iz} = \cos z + i \sin z.$$

Taking $z = \theta \in \mathbb{R}$ recovers the classical statement used in L01 to write polar form $re^{i\theta}$.

Solving for sine and cosine is immediate:

$$\cos z = \operatorname{Re}(e^{iz}) \quad (z \text{ real}), \quad \sin z = \operatorname{Im}(e^{iz}).$$

For complex z one must use the exponential expressions, not $\operatorname{Re} / \operatorname{Im}$.

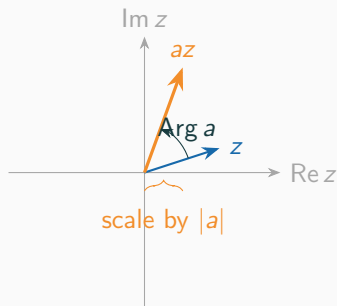
Why Euler explains multiplication

Polar form from L01 is now a theorem about the exponential:

$$z = r(\cos \theta + i \sin \theta) = re^{i\theta}.$$

Therefore

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}, \quad \frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}.$$



Addition formulae from Euler

Compute $e^{i(z+w)} = e^{iz}e^{iw}$ in two ways.

Left side: $\cos(z+w) + i\sin(z+w)$.

Right side:

$$(\cos z + i\sin z)(\cos w + i\sin w).$$

Equating real and imaginary parts (or using even/odd parts) yields

$$\sin(z+w) = \sin z \cos w + \cos z \sin w,$$

$$\cos(z+w) = \cos z \cos w - \sin z \sin w.$$

The same argument produces the double-angle formulae

$$\sin 2z = 2 \sin z \cos z, \quad \cos 2z = \cos^2 z - \sin^2 z.$$

Pythagoras still holds

From $e^{iz}e^{-iz} = 1$,

$$(\cos z + i \sin z)(\cos z - i \sin z) = 1,$$

hence the identity persists for all complex z :

$$\cos^2 z + \sin^2 z = 1.$$

Warning: this does *not* bound $|\sin z|$ or $|\cos z|$. Indeed

$$\cos^2 z + \sin^2 z = 1 \quad \text{but} \quad |\cos z|^2 + |\sin z|^2 = \cosh 2y$$

is typically much larger than 1.

Zeros of sine and cosine

$$\sin z = 0 \iff e^{iz} = e^{-iz} \iff e^{2iz} = 1 \iff z = n\pi, \quad n \in \mathbb{Z}.$$

Likewise $\cos z = 0$ iff $z = \frac{\pi}{2} + n\pi$.

No extra zeros

Complexification does *not* create new zeros of sin or cos. The zeros remain exactly the familiar real ones.

Hyperbolic functions: Sinh and cosh

Define

$$\cosh z = \frac{e^z + e^{-z}}{2}, \quad \sinh z = \frac{e^z - e^{-z}}{2}.$$

Then cosh is even, sinh is odd, and

$$\cosh^2 z - \sinh^2 z = 1, \quad (\sinh z)' = \cosh z, \quad (\cosh z)' = \sinh z.$$

Both are entire and satisfy $\cosh^2 z - \sinh^2 z = 1$.

Replacing z by iz in trig functions yields hyperbolic relationships:

$$\sin(iz) = i \sinh z, \quad \cos(iz) = \cosh z.$$

Conversely:

$$\sinh(iz) = i \sin z, \quad \cosh(iz) = \cos z.$$

Interpretation: Hyperbolic functions are trig functions rotated to the imaginary axis; this 90° rotation turns oscillation into exponential growth.

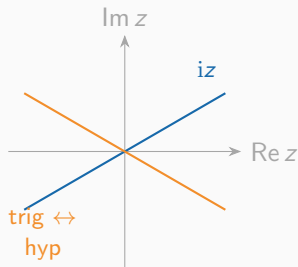
The dictionary $\text{trig} \leftrightarrow \text{hyp}$

Comparing definitions immediately gives

$$\begin{aligned}\cos(iz) &= \cosh z, & \cosh(iz) &= \cos z, \\ \sin(iz) &= i \sinh z, & \sinh(iz) &= i \sin z.\end{aligned}$$

Moral

Hyperbolic functions are trigonometric functions rotated onto the imaginary axis, and vice versa.



Analyticity summary

The exponential definitions give entire functions:

Function	Exponential form	Derivative	Cartesian form
$\sin z$	$(e^{iz} - e^{-iz})/2i$	$\cos z$	$\sin x \cosh y + i \cos x \sinh y$
$\cos z$	$(e^{iz} + e^{-iz})/2$	$-\sin z$	$\cos x \cosh y - i \sin x \sinh y$
$\sinh z$	$(e^z - e^{-z})/2$	$\cosh z$	$\sinh x \cos y + i \cosh x \sin y$
$\cosh z$	$(e^z + e^{-z})/2$	$\sinh z$	$\cosh x \cos y + i \sinh x \sin y$

Analyticity

Each power series has infinite radius of convergence. Hence $\sin z$, $\cos z$, $\sinh z$, and $\cosh z$ are analytic on all of $\mathbb{C}$; none requires a branch choice.

The identities: $\sin^2 z + \cos^2 z = 1$, $\cosh^2 z - \sinh^2 z = 1$, hold for every $z \in \mathbb{C}$.

Visualization of the complex exponential

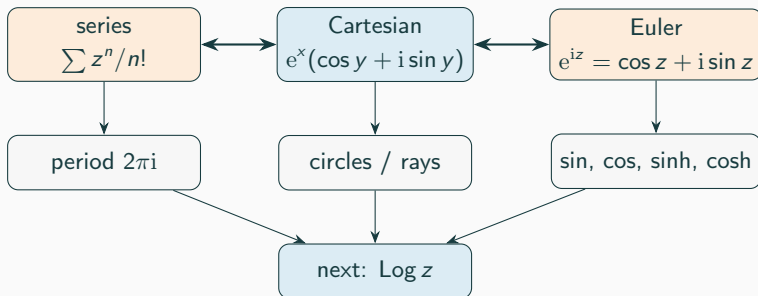
See the Wolfram notebook for details.

Practice

1. Cartesian form of $e^{4\pi i/3}$ and $e^{2-\pi i/3}$.
2. Find all θ with $e^{i\theta} = \sqrt{3}/2 + i/2$ and with $e^{i\theta} = -i$.
3. Derive $\sin(\theta_1 + \theta_2)$ from $e^{i(\theta_1 + \theta_2)}$.
4. Real and imaginary parts of $\sin\left(\frac{\sqrt{2}}{2}(1 + i)\right)$ and of $\cos(2 + i)$.
5. Real and imaginary parts of $\cos(-4 + 2\sqrt{3}i)$.

What we have learned & Concept map

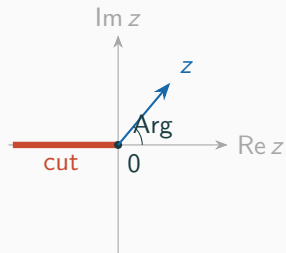
- $e^z = \sum z^n/n!$ is entire, never zero, and equals $e^x(\cos y + i \sin y)$
- $e^{z+w} = e^z e^w$ and $e^{z+2\pi i n} = e^z$
- vertical lines $\mapsto$ circles; horizontal lines $\mapsto$ rays
- $\sin z, \cos z$ extend by the same series; they are unbounded in the imaginary direction, yet keep the real zeros and $\sin^2 z + \cos^2 z = 1$
- hyperbolic functions are the imaginary-axis counterparts



The Complex Logarithm and Complex Powers

Key ideas

- Real logarithm as inverse of exp
- Solving $e^w = z$: infinitely many w
- Principal logarithm $\text{Log } z = \ln |z| + i \text{Arg } z$
- Branch cut of Log (negative real axis)
- Product rule with a $2\pi i n$ correction
- Complex powers $z^b = e^{b \log z}$



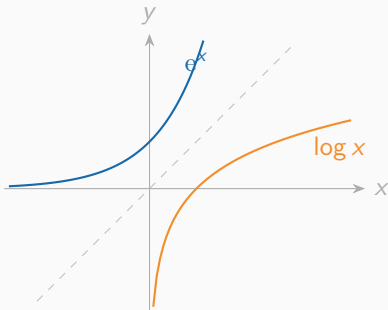
You will see how the logarithm is defined in the complex plane, and how powers z^b are defined for complex numbers. In general both have *branch cuts*.

Real exp and log as inverses

Recall the real logarithm

$\exp(\log x) = x$ for $x > 0$. The real domain of $\log$ is $(0, \infty)$.

Complex z is no longer restricted to the positive axis — but then $e^w = z$ has infinitely many solutions, so a convention is required.



- Graphs are reflections across $y = x$.
- $\log$ is defined only for $x > 0$.
- $\log(e^x) = x$ for all real x .
- $e^{\log x} = x$ for $x > 0$.

Over $\mathbb{C}$ the first identity will fail off a strip, because $\exp$ is $2\pi i$ -periodic.

Goal: invert the exponential

Given $z \in \mathbb{C}$, find w such that

$$e^w = z.$$

Obstruction

$\exp$ is *not injective*:

$$e^w = e^{w+2\pi i} = e^{w+4\pi i} = \dots$$

Infinitely many candidates $\dots, w - 2\pi i, w, w + 2\pi i, \dots$

A logarithm *function* must be single-valued, so we pick a *principal branch* — the same idea as for Arg .

Solving $e^w = z$

For $z = 0$ there is no solution: e^w never vanishes, so $\log 0$ is undefined. Otherwise write

$$z = |z| e^{i\theta}, \quad w = u + iv.$$

Then

$$e^w = e^{u+iv} = e^u e^{iv} = |z| e^{i\theta}.$$

Equating moduli and arguments:

$$u = \log |z| \quad (\text{real logarithm}), \quad v = \arg z.$$

The multi-valuedness of $\arg$ is inherited by $\log$. The complete set is

$$\log z = \ln |z| + i(\arg z)$$

or

$$\log z = \ln r + i(\theta + 2k\pi), \quad k \in \mathbb{Z}.$$

Definition of the (principal) logarithm

$$\text{Log } z = \ln |z| + i \text{Arg } z, \quad z \neq 0.$$

Using the principal argument $-\pi < \text{Arg } z \leq \pi$,

$$-\pi < \text{Im}(\text{Log } z) \leq \pi.$$

The image of Log is the horizontal strip $\{w : -\pi < \text{Im } w \leq \pi\}$.

Notation

Log (capital L) = principal value. The multi-valued logarithm is $\log z = \text{Log } z + 2\pi i n$, $n \in \mathbb{Z}$.

Two domains must be distinguished:

- a principal *value* is assigned for every $z \neq 0$;
- the principal *analytic branch* is defined on $\mathbb{C} \setminus (-\infty, 0]$.

On the cut plane, $-\pi < \text{Arg } z < \pi$ and $\text{Log } z$ is analytic.

Derivative of the logarithm

On a domain carrying one analytic branch $L(z)$,

$$e^{L(z)} = z.$$

Differentiating gives

$$e^{L(z)} L'(z) = 1.$$

Since $e^{L(z)} = z$,

$$\boxed{L'(z) = \frac{1}{z}}.$$

In particular,

$$(\operatorname{Log} z)' = \frac{1}{z}, \quad z \in \mathbb{C} \setminus (-\infty, 0].$$

Branch cut of Log: Two boundary values on the cut

Let $x < 0$. Approaching from the two sides gives

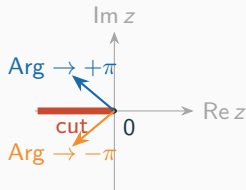
$$\lim_{y \downarrow 0} \text{Log}(x + iy) = \ln |x| + i\pi,$$

$$\lim_{y \uparrow 0} \text{Log}(x + iy) = \ln |x| - i\pi.$$

Their difference is $2\pi i$. Like Arg, the principal logarithm has a branch cut along the negative real axis, with branch point 0. Crossing the cut jumps Arg by 2π , hence Log jumps by $2\pi i$.

Meaning for later contour integration

A contour cannot cross the selected branch cut while using one analytic branch of Log z .

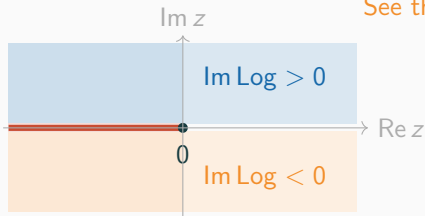


The cut is visible as a jump of the argument of Log z .

Modulus and argument of $\text{Log } z$

Coloring the plane by $\text{Arg}(\text{Log } z)$ and thinking of height $|\text{Log } z|$:

- singularity at 0 ($|\text{Log } z| \rightarrow \infty$)
- jump across $(-\infty, 0]$
- Log is continuous (and holomorphic) on $\mathbb{C} \setminus (-\infty, 0]$



See the Wolfram notebook for details.

See the Wolfram notebook for details.

Inverse identities

Always, for $z \neq 0$,

$$e^{\text{Log } z} = z,$$

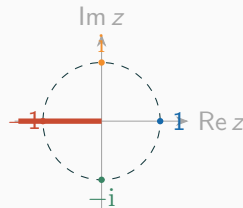
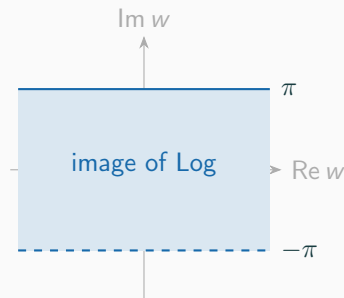
because exp cannot see the $2\pi i$ ambiguity. The other way around is *not* automatic:

$$\text{Log}(e^z) = z \quad \text{iff} \quad -\pi < \text{Im } z \leq \pi.$$

Special values of Log:

$$\text{Log } 1 = 0, \quad \text{Log } i = i\frac{\pi}{2}, \quad \text{Log}(-1) = i\pi, \quad \text{Log}(-i) = -i\frac{\pi}{2}.$$

And $\text{Log}(e^x) = x$ for real x . Off the positive real axis one must use Arg.



Caution — the product rule of Logarithm laws: what survives?

For real positive numbers, $\log(xy) = \log x + \log y$. Over $\mathbb{C}$,

$$\text{Log}(z_1 z_2) = \text{Log } z_1 + \text{Log } z_2 + 2\pi i n$$

for some integer $n \in \{-1, 0, 1\}$. The integer is *not* always zero!

Thus the familiar real logarithm law is not always an exact principal-value identity.

The same failure as for Arg

$\text{Log}(-1) + \text{Log}(-1) = i\pi + i\pi = 2\pi i$, but $\text{Log}((-1)(-1)) = \text{Log}(1) = 0$. Here $n = -1$.

Examples

$$\text{Log}(-1 + i\sqrt{3})$$

$|-1 + i\sqrt{3}| = 2$, and the point sits in the second quadrant:

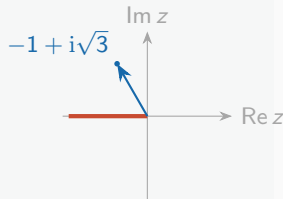
$$\text{Arg}(-1 + i\sqrt{3}) = \frac{2\pi}{3}.$$

Principal value:

$$\text{Log}(-1 + i\sqrt{3}) = \ln 2 + i\frac{2\pi}{3}.$$

All values of the multi-valued logarithm:

$$\log(-1 + i\sqrt{3}) = \ln 2 + i\frac{2\pi}{3} + 2\pi in, \quad n \in \mathbb{Z}.$$

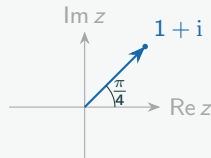


$$\text{Log}(1 + i)$$

$|1 + i| = \sqrt{2}$, $\text{Arg}(1 + i) = \pi/4$, so

$$\text{Log}(1 + i) = \ln \sqrt{2} + i\frac{\pi}{4} = \frac{1}{2} \ln 2 + i\frac{\pi}{4}.$$

Check: $e^{\text{Log}(1+i)} = \sqrt{2}e^{i\pi/4} = 1 + i$.



Differentiability of the logarithm

On the principal branch domain $\mathbb{C} \setminus (-\infty, 0]$, write $z = x + iy$ and

$$\operatorname{Log} z = u(x, y) + iv(x, y), \quad u = \ln |z| = \frac{1}{2} \ln(x^2 + y^2), \quad v = \operatorname{Arg} z = \operatorname{atan2}(y, x).$$

The partial derivatives are

$$u_x = \frac{x}{x^2 + y^2}, \quad u_y = \frac{y}{x^2 + y^2}, \quad v_x = -\frac{y}{x^2 + y^2}, \quad v_y = \frac{x}{x^2 + y^2}.$$

Thus $u_x = v_y$ and $u_y = -v_x$. The Cauchy–Riemann equations hold, and the partial derivatives are continuous on the slit plane. Therefore

$$(\operatorname{Log} z)' = u_x + iv_x = \frac{x - iy}{x^2 + y^2} = \frac{1}{z}.$$

The derivative is never zero on its analytic domain.

Summary: the logarithm $\text{Log } z$

Question	Answer
What is its value?	For $z = re^{i\theta} \neq 0$, the principal value is $\text{Log } z = \ln r + i\theta$, where $-\pi < \theta \leq \pi$.
What geometry does it create?	Circles $ z = r$ map to vertical lines $\text{Re } w = \ln r$; rays with fixed argument θ map to horizontal lines $\text{Im } w = \theta$.
What is its derivative?	$(\text{Log } z)' = 1/z$ on its analytic branch domain.
Where is it analytic?	The principal analytic branch is holomorphic on $\mathbb{C} \setminus (-\infty, 0]$; it has a branch point at 0 and a jump across the negative real axis.
Does it require a branch choice?	Yes. Solving $e^w = z$ gives $\log z = \ln z + i(\text{Arg } z + 2\pi k)$, $k \in \mathbb{Z}$. The principal branch selects $-\pi < \text{Im } w \leq \pi$.

Main message

Log is the analytic inverse of the exponential after a horizontal strip is chosen: modulus becomes real position and argument becomes imaginary position.

Definition of z^b

You are now in a position to define complex powers:

$$z^b = e^{b \log z}, \quad z \neq 0.$$

Because $\log$ is multi-valued, so is z^b in general.

- integer b : one value;
- rational non-integer b : finitely many values;
- irrational or non-real b : generally infinitely many values.

Principal branch of z^b

This course always uses the *principal* branch unless otherwise stated:

$$z^b = e^{b \operatorname{Log} z} = e^{b(\ln |z| + i \operatorname{Arg} z)}.$$

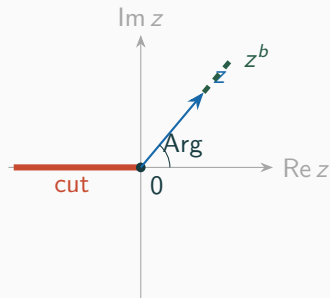
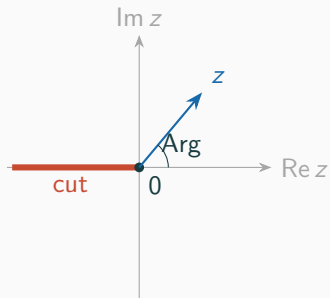
The branch cut of $z \mapsto z^b$ (for non-integer b) is the same cut as Log : the negative real axis, branch point 0. It is analytic on $\mathbb{C} \setminus (-\infty, 0]$ for the standard branch.

Branch dependence

Changing the logarithm branch by $2k\pi i$ multiplies the value by $e^{2k\pi i b}$. This factor is always 1 precisely when b is an integer.

Visualizing z^b

See the Wolfram notebook for details.



The cut of z^b is inherited from $\text{Log } z$. The branch choice of Log determines the value of z^b .

Derivative on a fixed branch

Let $f(z) = \exp(\alpha L(z))$, where L is an analytic logarithm branch. Then

$$f'(z) = \alpha L'(z) \exp(\alpha L(z)) = \frac{\alpha}{z} z^\alpha.$$

Therefore, on the same branch domain,

$$\boxed{\frac{d}{dz} z^\alpha = \alpha z^{\alpha-1}}.$$

The familiar formula survives only after a branch and its domain have been specified.

Example — all values of i^i

The logarithms of i are

$$\log i = i \left(\frac{\pi}{2} + 2k\pi \right), \quad k \in \mathbb{Z}.$$

Hence

$$i^i = \exp\left(i^2 \left[\frac{\pi}{2} + 2k\pi \right]\right) = \boxed{e^{-\pi/2 - 2k\pi}}.$$

Every value is positive real. The principal value is

$$\boxed{(i^i)_{\text{pr}} = e^{-\pi/2}}.$$

Example — all values of $(-1)^{2i}$

$$(-1)^{2i} = e^{2i \log(-1)} = e^{2i(\ln 1 + i(\pi + 2\pi n))} = e^{-2\pi - 4\pi n}.$$

The values are the positive reals

$$e^{-2\pi}, e^{-6\pi}, e^{2\pi}, e^{6\pi}, \dots$$

Principal value ($n = 0$): $e^{-2\pi} \approx 0.001867$.

Remark

A purely imaginary exponent of a negative base can produce a *real* number — a phenomenon with no real analogue. The branch index changes the magnitude, not merely the phase.

Example — the function z^{1+i}

On the principal branch,

$$z^{1+i} = e^{(1+i) \operatorname{Log} z}.$$

Writing $\operatorname{Log} z = \ln r + i\theta$,

$$(1+i)(\ln r + i\theta) = (\ln r - \theta) + i(\theta + \ln r),$$

so

$$|z^{1+i}| = r e^{-\theta}, \quad \operatorname{Arg}(z^{1+i}) = \theta + \ln r.$$

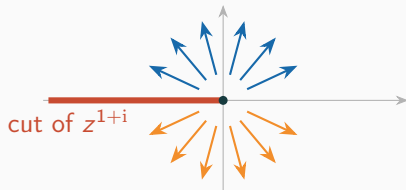
The modulus depends on the argument: walking around 0 changes $|z^{1+i}|$ exponentially. The cut of Log is inherited. The negative real axis is visible as a discontinuity inherited from $\operatorname{Log} z$.

- modulus and phase are coupled,
- changing the branch changes both.

See the Wolfram notebook for details.

Visualizing the cut of z^{1+i}

Stream / argument plots of z^{1+i} show the same negative-real cut as $\text{Log } z$. Crossing it jumps both modulus and argument.



See the Wolfram notebook for details.

Example — $(-2)^{1-i}$

Principal logarithm: $\text{Log}(-2) = \ln 2 + i\pi$. Then

$$(1 - i) \text{Log}(-2) = (1 - i)(\ln 2 + i\pi) = (\ln 2 + \pi) + i(\pi - \ln 2),$$

hence

$$(-2)^{1-i} = 2e^{\pi} (\cos(\pi - \ln 2) + i \sin(\pi - \ln 2)) = -2e^{\pi} (\cos \ln 2 - i \sin \ln 2).$$

Real and imaginary parts:

$$\text{Re} = -2e^{\pi} \cos \ln 2, \quad \text{Im} = 2e^{\pi} \sin \ln 2.$$

Example — $(-i)^{1+i}$

$\text{Log}(-i) = -i\pi/2$. Then

$$(1+i)\left(-i\frac{\pi}{2}\right) = \frac{\pi}{2} - i\frac{\pi}{2},$$

so

$$(-i)^{1+i} = e^{\pi/2} e^{-i\pi/2} = -ie^{\pi/2}.$$

Real part 0, imaginary part $-e^{\pi/2}$.

The celebrity i^i

$$i^i = e^{i \text{Log } i} = e^{i \cdot (i\pi/2)} = e^{-\pi/2} \in \mathbb{R}.$$

When is z^α single-valued?

- $\alpha \in \mathbb{Z}$: z^α is entire (or meromorphic at 0 for $\alpha < 0$). No cut.
- $\alpha = p/n \in \mathbb{Q}$ in lowest terms: n -valued, cut from 0 to ∞ (L05).
- $\alpha \notin \mathbb{Q}$: infinitely many values, still a cut from 0 to ∞ on the principal branch.

Standard convention

Always Log with cut $(-\infty, 0]$, and $z^\alpha = e^{\alpha \operatorname{Log} z}$.

Differentiability of z^b on a fixed branch

Let $L(z)$ be an analytic logarithm branch on a domain D and define

$$f(z) = z^b = \exp(bL(z)), \quad b \in \mathbb{C}.$$

Since $L'(z) = 1/z$ on D , the chain rule gives

$$\begin{aligned} f'(z) &= bL'(z) \exp(bL(z)) \\ &= \boxed{bz^{b-1}}. \end{aligned}$$

Thus the familiar power rule remains valid on every fixed analytic branch.

Important qualification

The formula applies on the branch domain D . A multi-valued power has no single complex derivative until one value of the logarithm is chosen.

For the principal power,

$$D = \mathbb{C} \setminus (-\infty, 0], \quad (z^b)' = bz^{b-1}.$$

Summary: the complex power z^b

Question	Answer
What is its value?	On a chosen branch, $z^b = \exp(b(\ln r + i\theta))$ for $z = re^{i\theta}$.
What geometry does it create?	If $b = \alpha + i\beta$, then $ z^b = r^\alpha e^{-\beta\theta}$ and $\arg(z^b) = \alpha\theta + \beta \ln r \pmod{2\pi}$. Thus radial scaling and rotation are coupled when b is non-real.
What is its derivative?	On a fixed analytic branch, $(z^b)' = bz^{b-1}$.
Where is it analytic?	For the principal branch, on $\mathbb{C} \setminus (-\infty, 0]$. If $b \in \mathbb{Z}_{\geq 0}$ it is entire; if $b \in \mathbb{Z}_{< 0}$ it is meromorphic on $\mathbb{C}$ with a pole at 0.
Does it require a branch choice?	Yes, unless b is an integer. The logarithm changes by $2\pi i k$, so the power changes by the factor $e^{2\pi i k b}$.

Main message

A complex power is an exponential of a logarithm: its calculus is ordinary on a fixed branch, while its global behavior is controlled by monodromy.

Practice

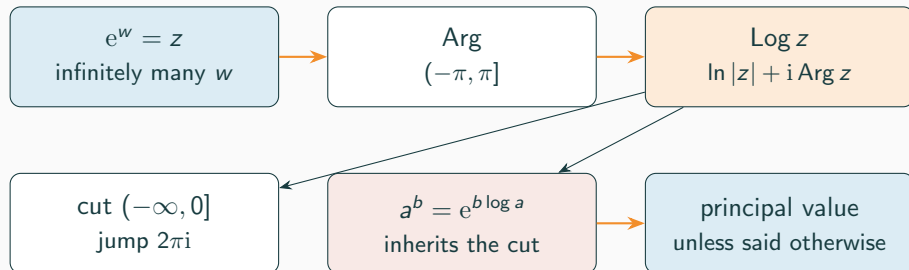
1. All values of $\log z$, and the principal one, for $z = i, -i, -1$.
2. Same for $z = 3 + i\sqrt{3}, \frac{1}{2} - i\frac{\sqrt{3}}{2}, -2 - 2i$.
3. Principal values of $3^{2i}, i^i, (-2)^{-i}$.
4. Principal values of $2^{1+i}, (-2)^{1-i/2}$.
5. Principal value of $(1 + i)^{i-1/2}$.

What we have learned & Concept map

- The complex logarithm has a branch cut. Its principal branch is

$$\text{Log } z = \ln |z| + i \text{Arg } z, \quad z \neq 0.$$

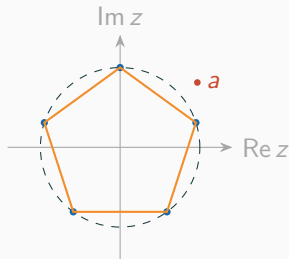
- Complex powers are defined by: $z^b = e^{b \log z}$, $z \neq 0$.
- In general, complex powers have a branch cut (the cut of $\log$).
- $e^{\text{Log } z} = z$ always; $\text{Log}(e^z) = z$ only on the strip $-\pi < \text{Im } z \leq \pi$.
- $\text{Log}(z_1 z_2) = \text{Log } z_1 + \text{Log } z_2 + 2\pi i n$.



Roots and Rational Powers

Key points

- The equation $z^n = a$ over $\mathbb{C}$
- n th roots as a regular n -gon
- A single-valued square root $\sqrt{z}$
- Branch points and branch cuts
- Rational powers of z



Real roots of $x^n = a$

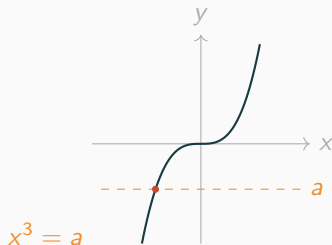
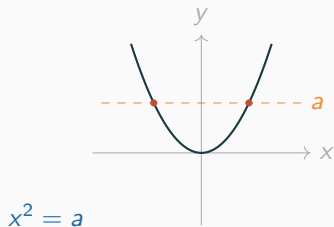
We first study the roots of $z^n = a$, which leads to a definition of the complex square root, and then to the argument function.

The obstruction

The square root cannot be made continuous on the whole plane, because of a *branch cut*.

First recall real analysis: for integer n , the equation $x^n = a$ has at most n real solutions.

Example: $x^2 = 4$ has $x = \pm 2$, and by convention $\sqrt{4} = +2$.



A quadratic has two, one, or no real roots according to the sign of a . A cubic always has at least one.

Over $\mathbb{C}$ there are always n roots: Polar form via Euler

Over $\mathbb{R}$, $z^2 = 1$ has solutions ± 1 , while $z^2 = -1$ has none. Over $\mathbb{C}$ the latter has $z = \pm i$.

Counting

For integer $n \geq 1$ and any $a \in \mathbb{C}$, the equation $z^n = a$ has *exactly* n complex solutions (counted with multiplicity). If $a = 0$ they all collapse to 0.

The extra room is polar form, plus the $2\pi i$ -periodicity of e^z . To find the n solutions, write (Euler's formula)

$$a = r(\cos \varphi + i \sin \varphi) = r e^{i\varphi}, \quad z = \rho(\cos \theta + i \sin \theta) = \rho e^{i\theta},$$

with $r = |a| > 0$ and $\rho = |z| > 0$. Then $z^n = a$ becomes

$$\rho^n e^{in\theta} = r e^{i\varphi}.$$

Equating moduli: $\rho^n = r$, hence $\rho = r^{1/n} > 0$.

The n solutions

Equating arguments, and using $e^{w+2\pi ik} = e^w$,

$$n\theta = \varphi + 2\pi k, \quad k \in \mathbb{Z},$$

so

$$\theta = \frac{\varphi + 2\pi k}{n}, \quad k = 0, 1, \dots, n-1.$$

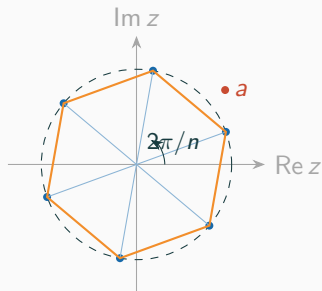
These n values of k give distinct θ modulo 2π , and therefore

$$z_k = r^{1/n} \exp\left(i \frac{\varphi + 2\pi k}{n}\right), \quad k = 0, 1, \dots, n-1.$$

Exactly n solutions — no more, no fewer.

Roots form a regular n -gon

All n roots lie on the circle of radius $r^{1/n}$, spaced by the angle $2\pi/n$: the vertices of a regular n -gon.



Solutions of $z^n = a$ as vertices of a regular n -gon.

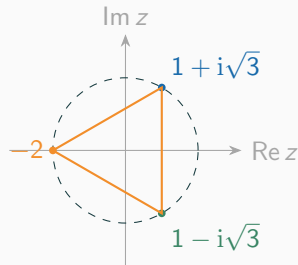
Example — cube roots of -8

Write $-8 = 8e^{i\pi}$. Then

$$z_k = 2 \exp\left(i \frac{\pi + 2\pi k}{3}\right), \quad k = 0, 1, 2.$$

$$z_0 = 2e^{i\pi/3} = 1 + i\sqrt{3}, \quad z_1 = 2e^{i\pi} = -2, \quad z_2 = 2e^{5i\pi/3} = 1 - i\sqrt{3}.$$

Each cubes to -8 , and the three points form an equilateral triangle.



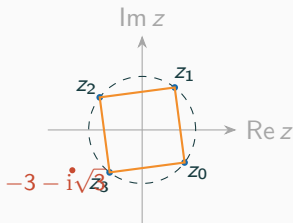
Example — $z^4 = -3 - i\sqrt{3}$

Modulus $r = \sqrt{9+3} = 2\sqrt{3}$. The point sits in the third quadrant, so

$$\text{Arg}(-3 - i\sqrt{3}) = -\pi + \frac{\pi}{6} = -5\pi/6.$$

$$z_k = (2\sqrt{3})^{1/4} \exp\left(i \frac{-5\pi/6 + 2\pi k}{4}\right), \quad k = 0, 1, 2, 3.$$

Four roots on a circle of radius $(2\sqrt{3})^{1/4}$, 90° apart (a square, rotated by -37.5°).

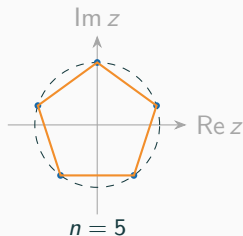


Roots of unity

Solutions of $z^n = 1$:

$$\omega_k = e^{2\pi i k/n}, \quad k = 0, \dots, n-1.$$

They lie on the unit circle. Their sum is zero, since the coefficient of z^{n-1} in $z^n - 1$ vanishes.



If z_0 is any particular n th root of a , the full set is

$$z_0, z_0\omega_1, \dots, z_0\omega_{n-1}.$$

Multiplying by roots of unity rotates the n -gon.

Two square roots, one function

$z^2 = a$ has two solutions in the complex plane. A square-root *function* must pick exactly one of them, otherwise it is not single-valued. The same issue already exists on $\mathbb{R}$: we define $\sqrt{x}$ as the *non-negative* (positive) solution of $w^2 = x$. That is a convention about what “square root” means, not a theorem.

Complex analogue

Choose, for each $z \in \mathbb{C}$, one of the two square roots according to a coherent criterion.

The standard convention

Require that $\sqrt{z}$ agrees with the real square root on the positive real axis, and is continuous in a neighborhood of that ray. Equivalently, using the principal argument (Wolfram Language, this course),

$$\sqrt{z} = \sqrt{|z|} e^{i \operatorname{Arg} z / 2}.$$

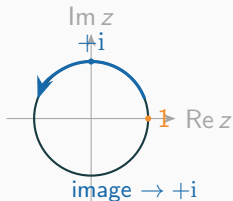
For $z = re^{i\theta}$ with $\theta = \operatorname{Arg} z \in (-\pi, \pi]$,

$$\sqrt{z} = \sqrt{r} e^{i\theta/2}, \quad \operatorname{Arg}(\sqrt{z}) = \frac{1}{2} \operatorname{Arg} z \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

Walking the unit circle

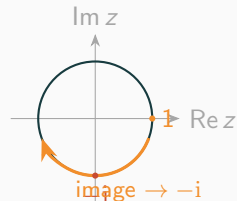
Points $e^{i\theta}$ map to $e^{i\theta/2}$.

Anticlockwise



$$0 \leq \theta \leq \pi$$

Clockwise



$$-\pi < \theta \leq 0$$

Approaching $\theta = \pi$ from above or below yields $+i$ or $-i$. Continuity across the negative real axis is impossible.

Why a square-root function needs a choice: Branch cuts and branch points

For every $z \neq 0$, the equation $w^2 = z$ has two solutions:

$$w = \pm \sqrt{r} e^{i\theta/2}, \quad z = r e^{i\theta}.$$

Continuing θ once around the origin changes

$$\sqrt{r} e^{i\theta/2} \longrightarrow \sqrt{r} e^{i(\theta+2\pi)/2} = -\sqrt{r} e^{i\theta/2}.$$

Consequence

No continuous single-valued square root exists on all of $\mathbb{C} \setminus \{0\}$.

The negative real axis is a *branch cut*, starting at the *branch point* 0.

Convention, not uniqueness

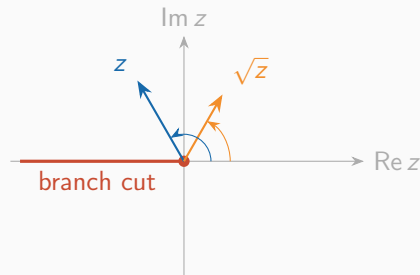
The cut could be placed along any ray from 0 to ∞ . One could put it along the positive imaginary axis and obtain a numerically different square root. This course never leaves the standard cut (negative real axis).

Principal square root and its branch cut

Choose $-\pi < \theta < \pi$. Then

$$\sqrt{z} = \sqrt{r} e^{i\theta/2}, \quad z \in \mathbb{C} \setminus (-\infty, 0].$$

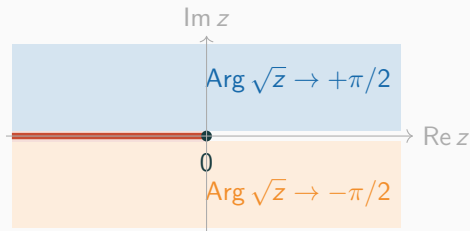
- branch point: $z = 0$,
- standard branch cut: negative real axis,
- argument range is halved to $(-\pi/2, \pi/2)$.



See the Wolfram notebook for details.

Visualizing $\sqrt{z}$ (argument jump)

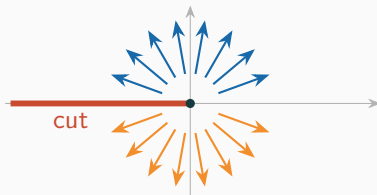
Coloring the plane by $\text{Arg } f(z)$ makes the cut a sudden hue change. For $f(z) = \sqrt{z}$ the argument jumps from $-\pi/2$ to $+\pi/2$ across $(-\infty, 0]$.



See the Wolfram notebook for details.

Vector / stream picture of $\sqrt{z}$

Stream lines of $\sqrt{z}$ refuse to cross the cut — a geometric picture of the jump. Near the positive real axis the field is continuous and agrees with the real square root.

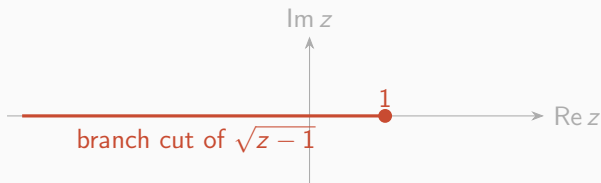


See the Wolfram notebook for details.

A shifted branch cut

Consider the principal branch $f(z) = \sqrt{z-1}$. The cut occurs when

$$z - 1 \in (-\infty, 0] \iff z \in (-\infty, 1].$$



Translation moves the branch point and the entire cut by the same amount.

See the Wolfram notebook for details.

Example — $\sqrt{(z+1)(z-1)}$

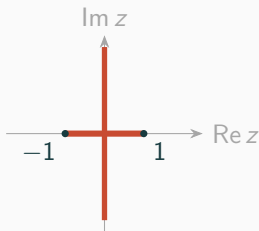
Cuts occur where the inside is real and ≤ 0 :

$$(z^2 - 1) \in (-\infty, 0].$$

Reducing over $\mathbb{C}$:

$$(\operatorname{Im} z = 0 \wedge -1 < \operatorname{Re} z < 1) \vee \operatorname{Re} z = 0.$$

In words: z purely imaginary, or z real between -1 and 1 . Branch points at $z = \pm 1$.



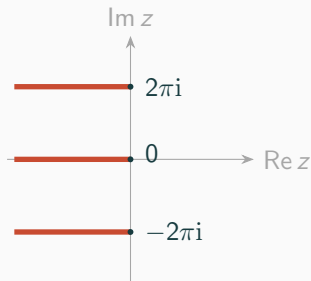
See the Wolfram notebook for details.

Example — $\sqrt{e^z - 1}$

Cuts where $e^z - 1$ is real and ≤ 0 . Since e^z never meets $(-\infty, 0]$, this is

$$e^z \in (0, 1] \iff \operatorname{Im} z = 2\pi n, \quad \operatorname{Re} z \leq 0, \quad n \in \mathbb{Z}.$$

Half-lines pointing left from the branch points $z = 2\pi in$.



See the Wolfram notebook for details.

Rational powers of z

The same reasoning applies to every non-integer rational power. If $q = p/n$ in lowest terms, z^q is n -valued, and a single-valued branch needs a cut from 0 to ∞ :

$$z^{p/n} = \exp\left(\frac{p}{n} \operatorname{Log} z\right) = |z|^{p/n} e^{ip \operatorname{Arg} z/n}.$$

In the principal convention the cut of every rational power of z is the negative real axis, with branch point 0.

If p/q is in lowest terms and $z \neq 0$, the possible values are

$$z^{p/q} = r^{p/q} e^{ip(\theta+2k\pi)/q}, \quad k = 0, 1, \dots, q-1.$$

In general there are q distinct values.

Example: values of $1^{1/3}$

$$1^{1/3} \in \{1, e^{2\pi i/3}, e^{4\pi i/3}\}.$$

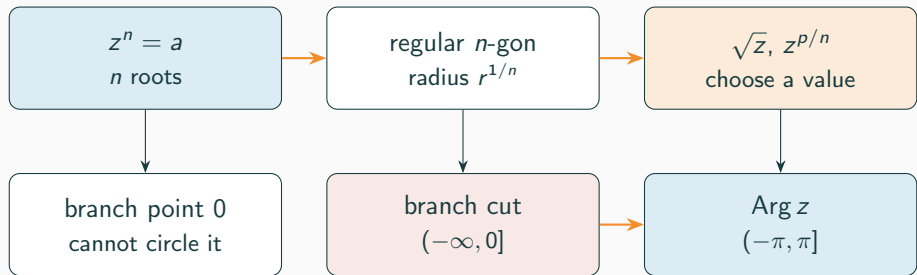
A principal value is obtained by using the principal argument. It inherits the branch cut of $\operatorname{Arg} z$.

Practice

1. Solutions of $z^4 = 1 + i$, and a plot.
2. Solutions of $z^3 = i$, and a plot.
3. Branch cuts of $f(z) = \sqrt{e^z + 1}$.
4. $\text{Arg } z$ for $z = 2 + i, 3 - 3i, -1 - \sqrt{3}i, -4 - 8i$.
5. $\text{Arg } z$ for $z = -1, 2i, -3 + \sqrt{3}i, -i$.

What we have learned

- $z^n = a$ has n complex roots — vertices of a regular n -gon.
- A single-valued z^q (q rational, not an integer) needs a cut from the branch point 0.
- Standard cut of $\sqrt{z}$ and of Arg : the negative real axis.
- $-\pi < \text{Arg } z \leq \pi$. Undefined at 0.
- Cuts of composite roots are the loci where the inside is real and ≤ 0 .



Inverse trigonometric and hyperbolic functions

Inverse trigonometric functions

The inverse trigonometric functions are obtained by solving the exponential forms of sine and cosine. On suitable principal branches,

$$\arcsin z = -i \operatorname{Log} \left(iz + \sqrt{1 - z^2} \right),$$

$$\arccos z = \frac{\pi}{2} - \arcsin z,$$

$$\arctan z = \frac{i}{2} [\operatorname{Log}(1 - iz) - \operatorname{Log}(1 + iz)].$$

Their derivatives are

$$(\arcsin z)' = \frac{1}{\sqrt{1 - z^2}}, \quad (\arccos z)' = -\frac{1}{\sqrt{1 - z^2}}, \quad (\arctan z)' = \frac{1}{1 + z^2},$$

wherever the selected branches are analytic.

Branch structure

Unlike $\sin$, $\cos$, and $\tan$, their inverses require logarithm and square-root branches. The branch points are ± 1 for $\arcsin$ and $\arccos$, and $\pm i$ for $\arctan$.

Inverse hyperbolic functions

The inverse hyperbolic functions have analogous logarithmic forms:

$$\operatorname{arsinh} z = \operatorname{Log}\left(z + \sqrt{z^2 + 1}\right),$$

$$\operatorname{arcosh} z = \operatorname{Log}\left(z + \sqrt{z - 1}\sqrt{z + 1}\right),$$

$$\operatorname{artanh} z = \frac{1}{2} [\operatorname{Log}(1 + z) - \operatorname{Log}(1 - z)].$$

Their derivatives are

$$(\operatorname{arsinh} z)' = \frac{1}{\sqrt{1 + z^2}}, \quad (\operatorname{arcosh} z)' = \frac{1}{\sqrt{z - 1}\sqrt{z + 1}}, \quad (\operatorname{artanh} z)' = \frac{1}{1 - z^2}.$$

Branch structure

These inverses are analytic only on domains where the displayed logarithm and square-root branches are consistent. Their branch points are $\pm i$ for arsinh and ± 1 for arcosh and artanh .

The real inverse functions are recovered on their usual real intervals.

A Unified View of Elementary Complex Functions

Single-valued entire functions

Function	Definition	Derivative	Key structure
e^z	$e^x(\cos y + i \sin y)$	e^z	period $2\pi i$, no zeros
$\sin z$	$\frac{e^{iz} - e^{-iz}}{2i}$	$\cos z$	zeros $k\pi$
$\cos z$	$\frac{e^{iz} + e^{-iz}}{2}$	$-\sin z$	zeros $\frac{\pi}{2} + k\pi$
$\sinh z$	$\frac{e^z - e^{-z}}{2}$	$\cosh z$	period $2\pi i$
$\cosh z$	$\frac{e^z + e^{-z}}{2}$	$\sinh z$	period $2\pi i$

All five functions are analytic on $\mathbb{C}$.

Branch-based functions

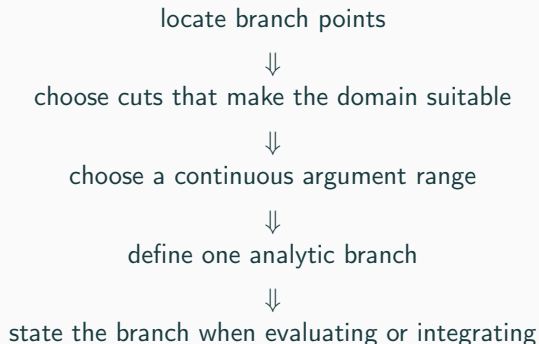
Object	Natural values	Principal choice	Analytic domain
$z^{1/n}$	n roots	$\exp(\text{Log } z/n)$	cut plane
$\arg z$	$\theta + 2k\pi$	$-\pi < \text{Arg } z \leq \pi$	not analytic
$\log z$	$\ln r + i(\theta + 2k\pi)$	$\text{Log } z$	cut plane
z^α	$\exp(\alpha \log z)$	$\exp(\alpha \text{Log } z)$	cut plane
$\arcsin, \arccos, \arctan$	inverse trig. values	Log and $\sqrt{\cdot}$ forms	branch cuts
$\text{arsinh}, \text{arcosh}, \text{artanh}$	inverse hyp. values	Log and $\sqrt{\cdot}$ forms	branch cuts

For the standard principal branch, the analytic cut plane is

$$\mathbb{C} \setminus (-\infty, 0].$$

Other cuts are possible when the contour geometry requires them.

A branch-choice workflow



Different cuts produce different branches, but each branch must be used consistently.

Why engineers care

Complex exponentials

Encode oscillation, damping, growth, and phase:

$$e^{(\sigma+i\omega)t}.$$

Trigonometric functions

Support frequency-domain descriptions and harmonic response.

Logarithms and powers

Linearize multiplicative scaling and describe fractional-order behavior.

Branch choices

Determine continuity, phase unwrapping, and whether a contour calculation is valid.

Summary

1. The exponential unifies growth and rotation and has period $2\pi i$.
2. Complex trigonometric and hyperbolic functions arise from exponentials and are generally unbounded off the real axis.
3. The equation $z^n = a$ produces regularly spaced roots.
4. Roots, arguments, logarithms, and non-integer powers are naturally multi-valued.
5. A branch cut permits one value to be chosen analytically on a suitable domain.
6. Derivative formulas remain valid only on the stated analytic domain.

Looking ahead to complex integration

L04 will integrate analytic functions along curves. This lecture supplies the functions and domains on which that theory operates.



A contour must not cross a singularity or the branch cut of the selected branch.

Next $\gg$ Lecture 04 — Integrals and Cauchy Theorems