

L04 — Integrals and Cauchy Theorems

Engineering Mathematics I

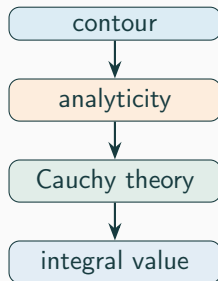
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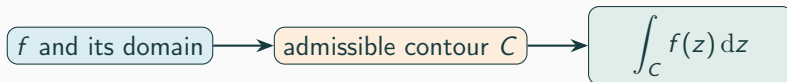
Road map

- Complex contours and line integrals
- Antiderivatives and path independence
- The Cauchy–Goursat theorem
- Contour deformation and applications
- Cauchy's integral formula
- Consequences of Cauchy theory
- Maximum modulus principle and minimum modulus principle
- Harmonic functions and orthogonal level curves



From analytic domains to contour integrals

L03 identified where elementary complex functions are analytic. We now ask what happens when those functions are accumulated along a curve.



The central issue

Does the answer depend on the detailed path, only on its endpoints, or only on which singularities the contour surrounds?

Complex Contours and Line Integrals

A contour is an oriented parametrized curve

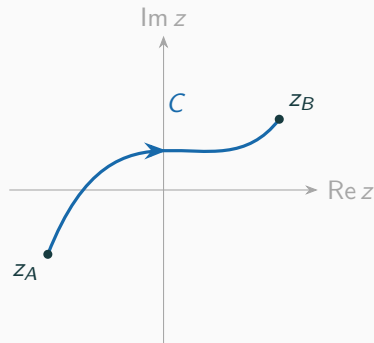
A contour C is described by

$$z = \gamma(t), \quad a \leq t \leq b,$$

with initial point $z_A = \gamma(a)$ and terminal point $z_B = \gamma(b)$.

Regularity used in this course

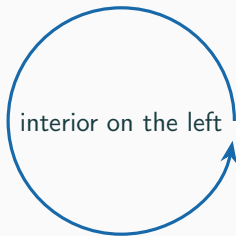
γ is continuously differentiable and $\gamma'(t) \neq 0$, except at finitely many joining points. Such a contour is *piecewise smooth*.



Orientation changes the sign

The reverse contour $-C$ is the same geometric curve traversed backwards:

$$\int_{-C} f(z) \, dz = - \int_C f(z) \, dz.$$



positive orientation



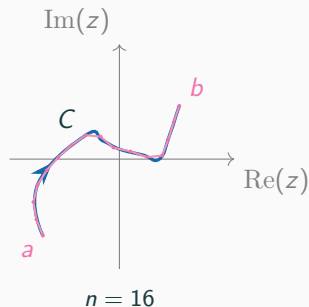
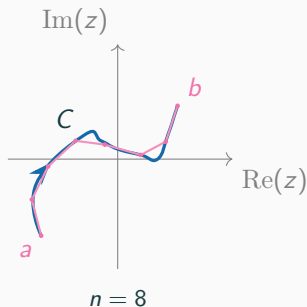
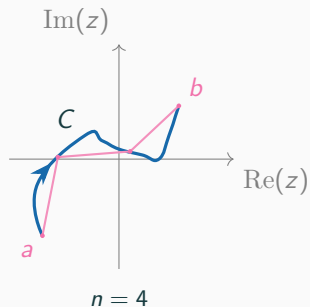
negative orientation

Counterclockwise orientation is positive unless stated otherwise.

Definition I: From Riemann sums to a contour integral

Divide $a = t_0 < t_1 < \dots < t_n = b$ and choose $\tau_k \in [t_{k-1}, t_k]$:

$$\int_C f(z) dz = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(\gamma(\tau_k)) (\gamma(t_k) - \gamma(t_{k-1})).$$



More subdivisions retain more information about the curve and the function.

Definition II: Computational definition by parametrization

If $z = \gamma(t)$ parametrizes C , then

$$\int_C f(z) \, dz = \int_a^b f(\gamma(t)) \gamma'(t) \, dt.$$

- Parametrize the contour with the correct orientation.
- Substitute $z = \gamma(t)$ and $dz = \gamma'(t) \, dt$.
- Evaluate an ordinary real-parameter integral.

Reparametrization invariance

Any smooth, orientation-preserving reparametrization gives the same value.

Definition III: Equivalent real-variable form

Write $f = u + iv$ and $dz = dx + i dy$. Then

$$f dz = (u dx - v dy) + i(v dx + u dy),$$

so

$$\int_C f dz = \int_C (u dx - v dy) + i \int_C (v dx + u dy).$$

This connects complex integration with vector calculus and explains why Green's theorem enters the proof of Cauchy's theorem.

Basic properties

For constants $\alpha, \beta \in \mathbb{C}$ and compatible oriented contours,

$$\int_C (\alpha f + \beta g) dz = \alpha \int_C f dz + \beta \int_C g dz,$$

$$\int_{C_1 \cup C_2} f dz = \int_{C_1} f dz + \int_{C_2} f dz,$$

$$\int_{-C} f dz = - \int_C f dz.$$

Piecewise contours

Break a complicated path into simple pieces and add their contributions in traversal order.

The ML estimate: bound without integrating

If $|f(z)| \leq M$ on a contour C of length L , then

$$\left| \int_C f(z) dz \right| \leq ML.$$

Indeed,

$$\left| \int_a^b f(\gamma(t)) \gamma'(t) dt \right| \leq \int_a^b |f(\gamma(t))| |\gamma'(t)| dt \leq ML.$$

Engineering use

An upper bound is often enough to prove that an error term or a large-radius contour contribution vanishes.

Worked example: integrate along a straight segment

Compute $\int_C z^2 dz$ from 0 to $2i$ along the straight path.

$$\gamma(t) = it,$$

$$0 \leq t \leq 2, \quad \gamma'(t) = i,$$

$$\int_C z^2 dz = \int_0^2 (it)^2 i dt = -i \int_0^2 t^2 dt = \boxed{-\frac{8}{3}i}.$$

Later, the antiderivative $F(z) = z^3/3$ will give the same result immediately.

A fundamental circular integral

Let $C : z = a + e^{it}$, $0 \leq t \leq 2\pi$, counterclockwise. For $n \in \mathbb{Z}$,

$$\oint_C \frac{1}{(z - a)^{n+1}} dz = \begin{cases} 2\pi i, & n = 0, \\ 0, & n \neq 0. \end{cases}$$

The exceptional kernel

The function $1/(z - a)$ records whether the contour winds around a . This kernel will drive Cauchy's integral formula.

Not every contour integral is path independent

Integrate $\bar{z}$ from 0 to $1 + i$.

Straight segment

$z = t(1 + i)$ gives

$$\int_C \bar{z} dz = \int_0^1 2t dt = 1.$$

Two segments

Along $0 \rightarrow 1 \rightarrow 1 + i$,

$$\int_C \bar{z} dz = \frac{1}{2} + \left(\frac{1}{2} + i\right) = 1 + i.$$

The integrand is continuous but not analytic. Path independence needs stronger structure.

Antiderivatives and Path Independence

Complex antiderivatives

A function F is an antiderivative of f in a domain Ω if

$$F'(z) = f(z) \quad (z \in \Omega).$$

If C joins z_A to z_B inside Ω , then

$$\int_C f(z) \, dz = F(z_B) - F(z_A).$$

Why it works

With $z = \gamma(t)$, the chain rule gives $\frac{d}{dt}F(\gamma(t)) = f(\gamma(t))\gamma'(t)$.

Fundamental theorem for complex line integrals

For continuous f on a domain Ω , the following are equivalent:

- f has an antiderivative in Ω ;
- $\int_C f(z) dz$ depends only on the endpoints of C ;
- $\oint_C f(z) dz = 0$ for every closed contour in Ω .



Worked example: direct calculation or antiderivative?

Compute $\int_C z^{-2} dz$ from 1 to i along the unit-circle arc.

Parametrize

$$z = e^{it}, 0 \leq t \leq \pi/2:$$

$$\int_0^{\pi/2} e^{-2it} i e^{it} dt = 1 + i.$$

Use $F(z) = -1/z$

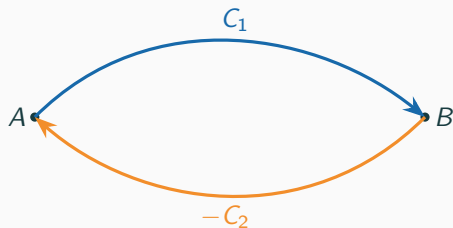
$$F(i) - F(1) = -\frac{1}{i} + 1 = 1 + i.$$

The second method works because the path lies in a domain where F is analytic.

Why closed integrals test path independence

Let C_1 and C_2 join the same endpoints. The closed contour $\Gamma = C_1 \cup (-C_2)$ satisfies

$$\oint_{\Gamma} f \, dz = \int_{C_1} f \, dz - \int_{C_2} f \, dz.$$



Thus closed integrals vanish exactly when paths with the same endpoints agree.

Why $1/(z - a)$ is different

Locally,

$$\frac{1}{z - a} = \frac{d}{dz} \log(z - a),$$

but one analytic branch of $\log(z - a)$ cannot cover a loop surrounding a .

$$a \notin \text{int } C \implies \oint_C \frac{dz}{z - a} = 0.$$

$$a \in \text{int } C \implies \oint_C \frac{dz}{z - a} = 2\pi i$$

for one positive winding.

The singularity prevents the loop from shrinking to a point.

A first method-selection rule

Before parametrizing, ask:

- Does the integrand have an antiderivative on a domain containing the path?
- Is the integral closed, and is the integrand analytic on and inside it?
- Does the contour enclose a singularity such as $1/(z - a)$?

Efficient practice

Parametrization is universal but often expensive. Analyticity and topology can reduce an integral to endpoint values, zero, or a standard kernel.

The Cauchy–Goursat Theorem

Notions from topology

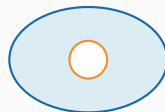
Curves

- A *simple* curve has no self-intersection.
- A *simple closed* curve returns to its starting point without crossing itself.



Regions

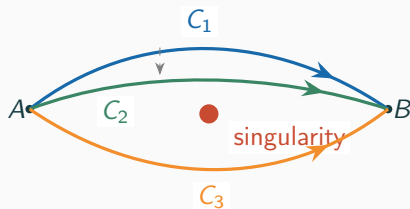
- A region is *connected* if it is one piece.
- It is *simply connected* if every closed curve can contract to a point within the region.



a hole prevents contraction

Homotopy: when can one path become another?

Two paths with the same endpoints are homotopic in Ω if one can be continuously deformed into the other without leaving Ω .



C_1 and C_2 are deformable above the singularity; C_3 lies in a different class.

Cauchy–Goursat theorem

Let f be analytic in a simply connected domain Ω . Then every closed contour $C \subset \Omega$ satisfies

$$\oint_C f(z) \, dz = 0.$$

Equivalently, if C_1 and C_2 have the same endpoints and are homotopic in the analytic domain of f , then

$$\int_{C_1} f(z) \, dz = \int_{C_2} f(z) \, dz.$$

Conditions matter

Analyticity is required throughout the region swept out by the deformation, not merely on the two curves.

Why the theorem is plausible

Write $f = u + iv$. Green's theorem applied to the real-variable form gives area integrals containing

$$v_x + u_y \quad \text{and} \quad u_x - v_y,$$

which vanish by the Cauchy–Riemann equations.

Cauchy versus Goursat

The Green's-theorem proof assumes continuous first partial derivatives. Goursat's refinement shows that complex differentiability alone is enough.

The long technical proof is compressed; the theorem conditions and geometric meaning are retained.

The theorem fails only when a hypothesis fails

For a closed contour C :

Analytic on and inside

$$\oint_C \frac{z-1}{(z+2)^2} dz = 0$$

whenever -2 lies outside C .

A pole lies inside

$$\oint_C \frac{dz}{z-a}$$

is not forced to be zero when a is enclosed.

Always locate singularities before invoking Cauchy's theorem.

Analytic functions have antiderivatives locally

If f is analytic in a simply connected domain Ω , fix $z_0 \in \Omega$ and define

$$F(z) = \int_{z_0}^z f(w) \, dw.$$

Cauchy's theorem makes this path independent, and $F'(z) = f(z)$.

Global warning

In a multiply connected domain an analytic function need not have a global antiderivative; $1/z$ on $\mathbb{C} \setminus \{0\}$ is the standard example.

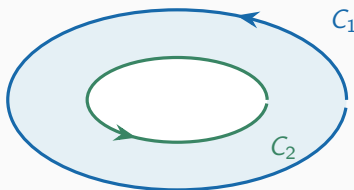
Contour Deformation and Applications

Deformation principle

If f is analytic throughout the region between two positively oriented closed contours C_1 and C_2 , then

$$\oint_{C_1} f(z) dz = \oint_{C_2} f(z) dz.$$

f analytic in the shaded region

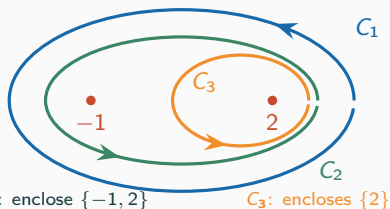


The integral depends on the enclosed singularity structure, not the exact shape.

Worked example: when do contour integrals agree?

Let

$$f(z) = \frac{z-1}{(z-2)(z+1)}.$$



Partial fractions

$$f(z) = \frac{1/3}{z-2} + \frac{2/3}{z+1}$$

$$\oint_{C_1} f \, dz = 2\pi i, \quad \oint_{C_3} f \, dz = \frac{2\pi i}{3}$$

C_1 and C_2

Same enclosed singularities.

$$\int_{C_1} f \, dz = \int_{C_2} f \, dz$$

by deformation principle.

C_3 is different

Encloses only $z = 2$.

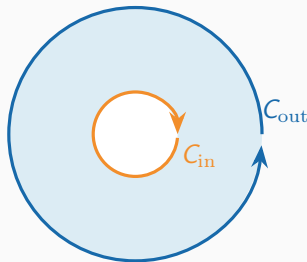
$$\oint_{C_3} f \, dz \neq \oint_{C_1} f \, dz$$

No continuous deformation exists without crossing $z = -1$.

Boundary orientation in a region with holes

For an analytic function on a region with one hole, the positive boundary has an outer counterclockwise contour and an inner clockwise contour:

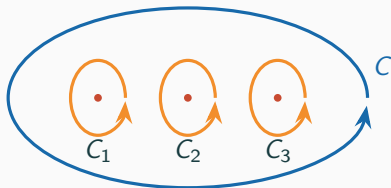
$$0 = \oint_{\partial\Omega} f \, dz = \oint_{C_{\text{out}}} f \, dz - \oint_{C_{\text{in}}} f \, dz.$$



Several holes: split one contour into many

If f is analytic in the region between C and disjoint inner contours $C_1, \dots, C_m$, then

$$\oint_C f(z) dz = \sum_{k=1}^m \oint_{C_k} f(z) dz.$$



This is the geometric precursor of residue calculus in L06.

Worked example: isolate two singularities

Compute

$$\oint_{|z|=5} \left(\frac{1}{z} + \frac{1}{z-3} \right) dz.$$

Both singularities 0 and 3 lie inside. Deform into two small positive circles:

$$\begin{aligned} \oint_{|z|=5} \left(\frac{1}{z} + \frac{1}{z-3} \right) dz &= \oint_{C_0} \frac{dz}{z} + \oint_{C_3} \frac{dz}{z-3} \\ &= 2\pi i + 2\pi i = \boxed{4\pi i}. \end{aligned}$$

Only crossed singularities can change the value

For

$$f(z) = \frac{z}{z^2 - 1},$$

the annulus $4 \leq |z| \leq 5$ contains no singularity. Therefore

$$\oint_{|z|=4} f(z) \, dz = \oint_{|z|=5} f(z) \, dz.$$

Deformation rule

Stretch, compress, or reshape a contour freely provided no singularity is crossed and the contour remains inside the analytic domain.

Deform outward and use an estimate

Suppose a contour expands to $|z| = R$ without crossing a singularity and

$$|f(z)| = O(R^{-p}), \quad p > 1.$$

The ML estimate gives

$$\left| \oint_{|z|=R} f(z) dz \right| \leq 2\pi R O(R^{-p}) \rightarrow 0.$$

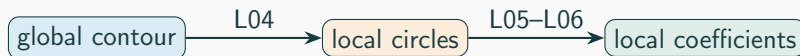
Deformation then shows that the original integral is zero.

Useful pattern

Topology moves the contour to a convenient location; estimation finishes the calculation.

What deformation does not yet automate

With several complicated singularities, deformation reduces one contour to local circles, but each local integral still has to be evaluated.

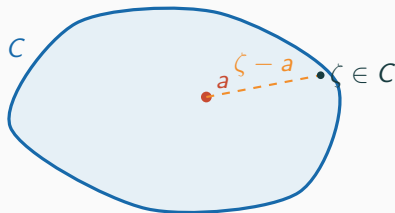


Laurent series and residues will later supply the systematic local coefficient.

Cauchy's Integral Formula

The Cauchy kernel reproduces an analytic function

Let a lie inside a positively oriented simple closed contour C .



The kernel $1/(\zeta - a)$ concentrates boundary information at the interior point a .

Cauchy's integral formula

If f is analytic on and inside C , and a is inside C , then

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - a} d\zeta.$$

Equivalently,

$$\oint_C \frac{f(\zeta)}{\zeta - a} d\zeta = 2\pi i f(a).$$

Pattern recognition

Identify a , verify that it is inside C , and check that the numerator is analytic on and inside the contour.

Proof idea I: separate the singular part

Write

$$\frac{f(\zeta)}{\zeta - a} = \frac{f(\zeta) - f(a)}{\zeta - a} + \frac{f(a)}{\zeta - a}.$$

The second term is known:

$$\oint_C \frac{f(a)}{\zeta - a} d\zeta = 2\pi i f(a).$$

The first quotient has a removable singularity because

$$\lim_{\zeta \rightarrow a} \frac{f(\zeta) - f(a)}{\zeta - a} = f'(a).$$

Proof idea II: shrink and estimate

Deform C to $|\zeta - a| = r$. If

$$g(\zeta) = \frac{f(\zeta) - f(a)}{\zeta - a}$$

is bounded by M near a , then

$$\left| \oint_{|\zeta - a| = r} g(\zeta) d\zeta \right| \leq M(2\pi r) \longrightarrow 0.$$

Therefore only the standard kernel remains.

Three ideas combined

Removable singularity + contour deformation + ML estimate.

Worked example: read the value directly

Let C enclose π . Compute

$$I = \oint_C \frac{\cos z}{z - \pi} dz.$$

Here $f(z) = \cos z$ is entire and $a = \pi$, so

$$I = 2\pi i f(\pi) = 2\pi i \cos \pi = \boxed{-2\pi i}.$$

No parametrization of C is needed.

Worked example: first check whether the pole is enclosed

Compute over $|z| = 2$:

$$\oint_{|z|=2} \frac{e^z}{z+1} dz.$$

Since $a = -1$ is inside and e^z is entire,

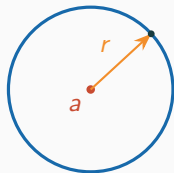
$$\oint_{|z|=2} \frac{e^z}{z - (-1)} dz = 2\pi i e^{-1} = \boxed{\frac{2\pi i}{e}}.$$

If the contour did not enclose -1 , Cauchy's theorem would give zero.

Mean-value property for analytic functions

On $\zeta = a + re^{i\theta}$, Cauchy's formula becomes

$$f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta}) d\theta.$$



center value
= boundary average

Analyticity couples interior values so strongly that boundary data determine the center.

Cauchy's formula for derivatives

Differentiating under the integral sign gives, for $n \geq 0$,

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - a)^{n+1}} d\zeta.$$

Hence one complex derivative implies derivatives of every order.

Recognition pattern

A denominator $(z - a)^{n+1}$ requests the n th derivative of the analytic numerator at a .

Worked example: a second-order denominator

For a contour enclosing 1,

$$I = \oint_C \frac{\cos z}{(z-1)^2} dz.$$

Since $(z-1)^2$ corresponds to $n = 1$,

$$I = 2\pi i \left. \frac{d}{dz} \cos z \right|_{z=1} = \boxed{-2\pi i \sin 1}.$$

The exponent determines the derivative order: $n = (\text{power}) - 1$.

Worked example: isolate the analytic numerator

Let C enclose the origin. Compute

$$I = \oint_C \frac{2z^2 - 1}{z^3} dz.$$

Set $f(z) = 2z^2 - 1$ and use $n = 2$:

$$I = \frac{2\pi i}{2!} f''(0) = \frac{2\pi i}{2} (4) = \boxed{4\pi i}.$$

Apply the formula to the numerator, not to the entire singular integrand.

Consequences of Cauchy Theory

Liouville's theorem

Theorem

Every bounded entire function is constant.

If $|f(z)| \leq M$ on $\mathbb{C}$, the Cauchy estimate on a circle of arbitrary radius R gives

$$|f'(a)| \leq \frac{M}{R}.$$

Letting $R \rightarrow \infty$ yields $f'(a) = 0$ for every a , hence f is constant.

Complex rigidity

Globally bounded analytic behavior is far more restrictive in $\mathbb{C}$ than in $\mathbb{R}$.

Why complex sine does not contradict Liouville

On the real axis, $|\sin x| \leq 1$. For $z = x + iy$,

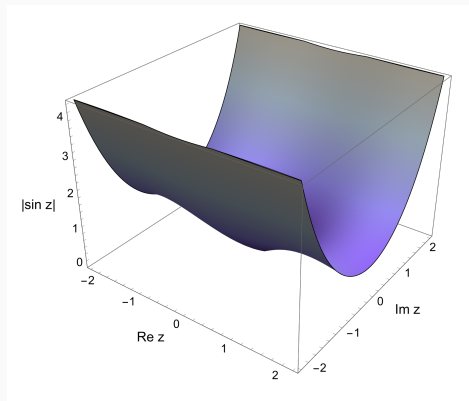
$$\sin z = \sin x \cosh y + i \cos x \sinh y.$$

Along the imaginary axis,

$$|\sin(iy)| = |\sinh y| \rightarrow \infty.$$

Liouville applies globally

The complex sine is entire, but it is not bounded on $\mathbb{C}$.



The growth of $|\sin z|$ away from the real axis.

Cauchy estimates

Suppose f is analytic on and inside $|z - a| = R$ and $|f(z)| \leq M_R$ on the circle. Then

$$|f^{(n)}(a)| \leq \frac{n! M_R}{R^n}.$$

The derivative formula and ML estimate give

$$|f^{(n)}(a)| \leq \frac{n!}{2\pi} \frac{M_R}{R^{n+1}} (2\pi R).$$

These estimates control derivatives, Taylor coefficients, and approximation errors.

Fundamental Theorem of Algebra: an analytic proof

L01 established that a degree- n polynomial has n complex roots counting multiplicity. Suppose a nonconstant polynomial P had no zero. Then

$$g(z) = \frac{1}{P(z)}$$

would be entire. Moreover, $g(z) \rightarrow 0$ as $|z| \rightarrow \infty$, and continuity bounds it on every finite disk. Hence g is bounded on $\mathbb{C}$. Liouville's theorem would make g constant, a contradiction. Therefore P has a root; factorization completes the result.

Four roots remain four as coefficients vary

For

$$P(z) = z^4 + az^3 + bz^2 + 1,$$

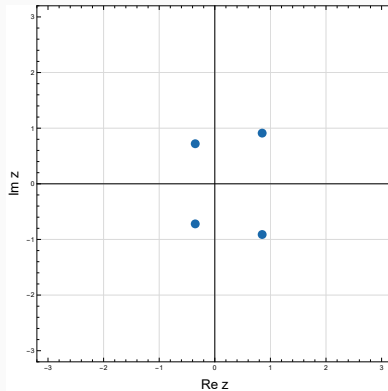
the locations and number of real roots may change with a and b , but four complex roots remain when multiplicity is counted.

Live demonstration

Vary a and b and track the roots continuously in the complex plane.

See the [Wolfram notebook](#) for details.

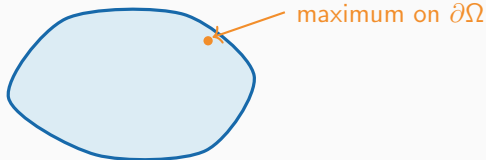
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Maximum modulus principle

If f is analytic and nonconstant in a connected open domain Ω , then $|f|$ has no local maximum in the interior. If f is also continuous on the boundary of a bounded domain, then

$$\max_{z \in \overline{\Omega}} |f(z)| = \max_{z \in \partial\Omega} |f(z)|.$$



no interior maximum of $|f|$

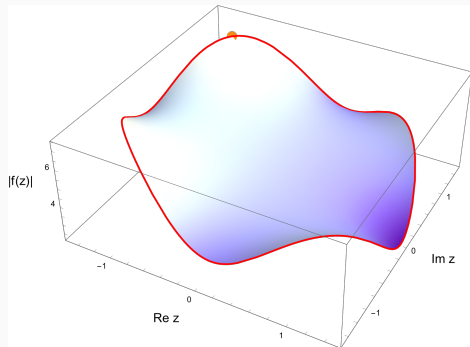
A boundary maximum in three dimensions

For

$$f(z) = \frac{z^4}{4} + z - 5,$$

the surface height is $|f(z)|$ over a disk.

- red curve: boundary values;
- orange point: largest value on the disk;
- no interior peak exceeds the boundary.



Minimum modulus principle and zeros

If f is analytic and nonzero in Ω , then $1/f$ is analytic. Applying the maximum modulus principle to $1/f$ gives

$$\min_{z \in \overline{\Omega}} |f(z)| = \min_{z \in \partial\Omega} |f(z)|$$

for a nonconstant f continuous on the closure.

Why the nonzero condition is essential

If $f(a) = 0$ inside, then $|f|$ has the interior minimum 0 and $1/f$ is not analytic there.

Harmonic Functions and Orthogonal Level Curves

Harmonic function

A function $u(x, y)$ is *harmonic* if it satisfies Laplace's equation

$$u_{xx} + u_{yy} = 0.$$

Physical models

- Electrostatics: the potential $V(x, y, z)$ is harmonic.
- Heat exchange: in thermal equilibrium $\partial T / \partial t = 0$, so the steady temperature $T(x, y, z)$ is harmonic.

Link to analytic functions

If $f(z) = u(x, y) + iv(x, y)$ is analytic, then both u and v are harmonic — a direct consequence of the Cauchy–Riemann equations.

$f = u + iv$ analytic

→ u harmonic

→ v harmonic

Level curves of harmonic conjugates are orthogonal

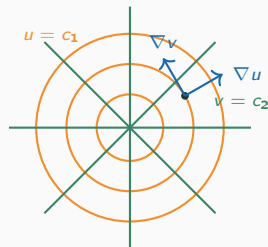
Theorem

If $f(z) = u(x, y) + iv(x, y)$ is analytic and $f'(z) \neq 0$, then the level curves $u(x, y) = c_1$ and $v(x, y) = c_2$ intersect at right angles.

Proof. The gradients $\nabla u = (u_x, u_y)$ and $\nabla v = (v_x, v_y)$ are normal to the two level curves. Their dot product is

$$(\nabla u) \cdot (\nabla v) = u_x v_x + u_y v_y \stackrel{\text{C-R}}{=} u_x(-u_y) + u_y u_x = 0.$$

Since $f'(z) \neq 0$, neither gradient vanishes, so the curves are orthogonal.



Field interpretation

Equipotential lines (u) and streamlines/field lines (v) always cross orthogonally.

Where orthogonality can fail: $f'(z) = 0$

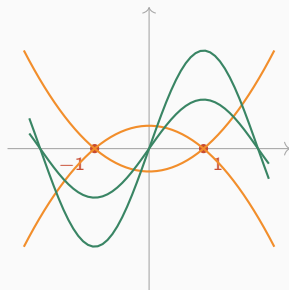
For $f(z) = z^3 - 3z + 2$,

$$u(x, y) = 2 - 3x + x^3 - 3xy^2, \quad v(x, y) = -3y + 3x^2y - y^3.$$

- The level curves of u and v interlace orthogonally almost everywhere.
- But $f'(z) = 3z^2 - 3$ vanishes at $z = \pm 1$: there the gradients $\nabla u = \nabla v = \mathbf{0}$, so the orthogonality argument has no content.

Condition check

Orthogonality is guaranteed only where $f'(z) \neq 0$; at critical points level curves may touch or cross at other angles.



level curves meeting at $f'(z) = 0$

Finding the harmonic conjugate of a harmonic function

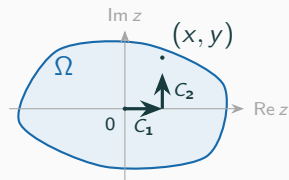
Given u harmonic on a *simply connected* domain Ω , we seek v with

$$\begin{cases} v_x = -u_y, \\ v_y = u_x, \end{cases} \quad \text{i.e. } f = u + iv \text{ analytic.}$$

Integrating along the two-segment path $C_1 \cup C_2$:

$$v(x, 0) = - \int_0^x u_y(x', 0) dx',$$

$$v(x, y) = v(x, 0) + \int_0^y u_x(x, y') dy'.$$



If Ω is *not* simply connected, the procedure may fail to define v uniquely — cf.

$u = \log \sqrt{x^2 + y^2}$ and the multi-valued conjugate $\arg z$.

Worked example: conjugate of $u = x^3 - 3xy^2 + 4xy$

Step 1. Check harmonicity:

$$u_{xx} + u_{yy} = 6x - 6x = 0. \checkmark$$

Step 2. Cauchy–Riemann data:

$$u_x = 3x^2 - 3y^2 + 4y = v_y,$$

$$u_y = -6xy + 4x = -v_x.$$

Step 3. Integrate along the axes:

$$v(x, 0) = -\int_0^x u_y(x', 0) dx' = -\int_0^x 4x' dx' = -2x^2,$$

$$v(x, y) = -2x^2 + \int_0^y (3x^2 - 3y'^2 + 4y') dy',$$

$$v(x, y) = -2x^2 + 3x^2y + 2y^2 - y^3.$$

Verification

Direct computation gives $\nabla^2 v = 0$, and $v_y = 3x^2 - 3y^2 + 4y = u_x$, $v_x = -4x + 6xy = -u_y$: both Cauchy–Riemann equations hold. Hence $f(z) = u + iv$ is analytic (f differs from $z^3 + 2iz^2$ by at most a constant).

Harmonic mean-value property

If u is harmonic on a region containing $|z - z_0| \leq r$, then

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta.$$

This follows by taking the real or imaginary part of the analytic mean-value formula.

Physical interpretation

In a steady-state temperature or potential field, the center value equals the average around every sufficiently small surrounding circle.

Harmonic maximum and minimum principles

If a harmonic function u is nonconstant in a bounded connected domain and continuous on its closure, then

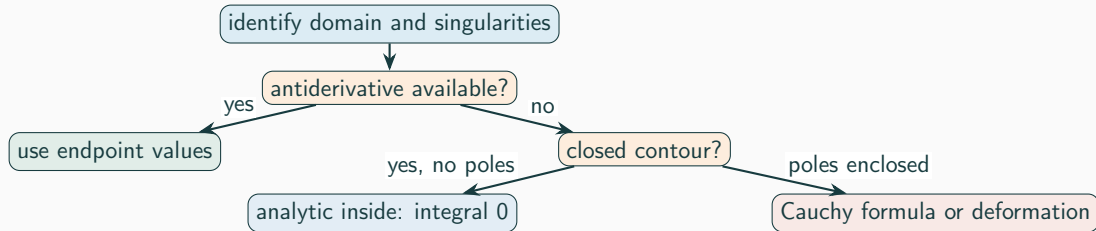
$$\max_{\bar{\Omega}} u = \max_{\partial\Omega} u, \quad \min_{\bar{\Omega}} u = \min_{\partial\Omega} u.$$

Engineering consequence

A source-free steady field cannot create an interior hot spot, cold spot, voltage maximum, or voltage minimum; extrema are imposed by the boundary.

Harmonic conjugates and field-line geometry are deliberately reserved for L07.

A practical workflow for contour integrals



Parametrize only when structural methods do not apply or when the definition is being tested.

Summary: What we have learned

- Contour integrals are computed by parametrization and controlled by orientation.
- Antiderivatives make integrals path independent.
- Cauchy–Goursat turns analyticity into strong integral invariance.
- Contour deformation replaces complicated paths with convenient local contours.
- Cauchy's formula reconstructs values and all derivatives from boundary data.
- Liouville, algebraic, and maximum principles reveal analytic rigidity.

Looking ahead to series

Cauchy's formula writes

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z} d\zeta.$$

Expanding the kernel around a center converts this global boundary representation into local coefficients:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

L05 develops convergence disks, Taylor series, Laurent series, zeros, and isolated singularities.

Next $\gg$ Lecture 05 — Series