

L05 — Series

Engineering Mathematics I

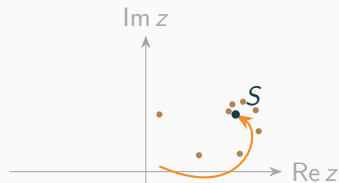
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Road map

- Power series
- Taylor series
- Laurent series
- Holomorphic and meromorphic functions



partial sums approach
the value of a series

Why series? From exact formulas to computable models

L04 gave the exact representation

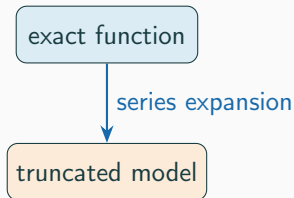
$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z} d\zeta.$$

Expanding the kernel converts global contour information into a **local algebraic model**:

$$f(z) \approx a_0 + a_1(z - z_0) + \cdots + a_N(z - z_0)^N.$$

Engineering viewpoint

A truncated series replaces a difficult function by a computable polynomial, with accuracy controlled by the convergence region and the neglected remainder.



Power series

Series and partial sums

A **series** is an infinite sum

$$c_0 + c_1 + c_2 + \cdots = \sum_{n=0}^{\infty} c_n.$$

Its **partial sums** are

$$s_N = c_0 + c_1 + \cdots + c_N = \sum_{n=0}^N c_n.$$

Convergence

The series converges to C if $\lim_{N \rightarrow \infty} s_N = C$.

Thus convergence is a statement about the sequence (s_N) , not merely about the terms c_n .

A necessary condition for convergence

If the series $\sum_{n=0}^{\infty} c_n$ converges, then

$$\boxed{c_n \longrightarrow 0}.$$

Indeed, since $c_n = s_n - s_{n-1}$ and both partial sums approach the same limit,

$$c_n = s_n - s_{n-1} \longrightarrow C - C = 0.$$

Quick divergence check

If c_n does not approach 0, then $\sum c_n$ diverges. The converse is false: $c_n \rightarrow 0$ alone does not guarantee convergence.

Absolute convergence

The series $\sum c_n$ is **absolutely convergent** if

$$\sum_{n=0}^{\infty} |c_n| < \infty.$$

Basic implication

Every absolutely convergent complex series is convergent.

Indeed, for $q > p$,

$$\left| \sum_{n=p}^q c_n \right| \leq \sum_{n=p}^q |c_n|,$$

so the partial sums form a Cauchy sequence in $\mathbb{C}$.

Definition of a power series

A **power series** centered at z_0 is

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad a_n, z, z_0 \in \mathbb{C}.$$

- z : the **variable** that changes from point to point.
- z_0 : the **center** of the expansion (center of the convergence disk).
- a_n : the **coefficients** (fixed complex constants, no z)

The convergence theorem

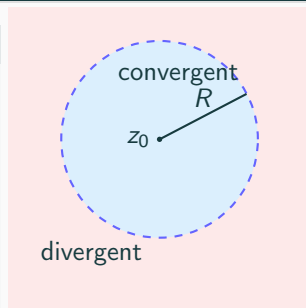
Theorem

For every power series there is a number $R \in [0, \infty]$ such that

- $|z - z_0| < R \Rightarrow$ absolute convergence
- $|z - z_0| > R \Rightarrow$ divergence
- $|z - z_0| = R$: each boundary point requires a separate test
- In $|z - z_0| < R$, $f(z)$ is analytic and can be differentiated term by term.

The number R is the **radius of convergence**.

- The blue disk is the region of absolute convergence.
- The pink region is the region of divergence.
- The dashed circle requires separate analysis.



Region of convergence of a power series.

Worked example: the geometric series

For $z \neq 1$,

$$1 + z + \cdots + z^N = \frac{1 - z^{N+1}}{1 - z}.$$

Therefore

$$\boxed{\sum_{n=0}^{\infty} z^n = \frac{1}{1 - z}}, \quad |z| < 1,$$

For $|z| < 1$, the remainder satisfies

$$\left| \frac{z^{N+1}}{1 - z} \right| \rightarrow 0,$$

so the radius of convergence is $R = 1$. On $|z| = 1$, the terms z^n do not approach zero, so the series diverges.

Three familiar power series

The notebook recalls three functions already defined by power series:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!},$$

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!},$$

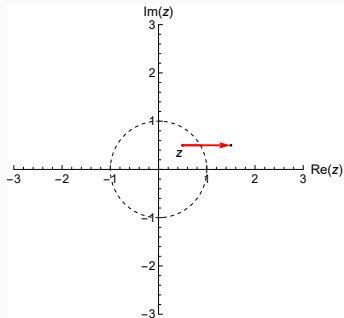
$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}.$$

Each has infinite radius of convergence.

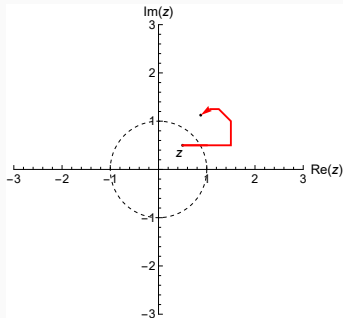
Geometric interpretation

Complex addition is vector addition, so successive terms can be drawn head to tail.

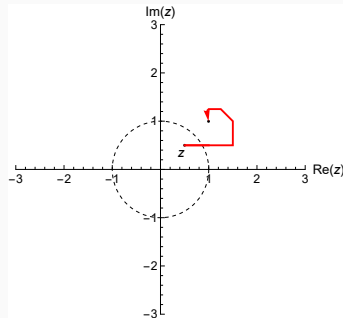
Partial sums as vectors



$N = 1$



$N = 5$



$N = 15$

Convergence of the geometric series for $z = \frac{1}{2} + \frac{1}{2}i$.

For a series $\sum c_n$, suppose

$$L = \lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} \right|$$

exists. Then $L < 1$ gives absolute convergence, $L > 1$ gives divergence, and $L = 1$ is inconclusive.

Example: For the geometric series, $c_n = z^n$, so

$$L = \lim_{n \rightarrow \infty} \left| \frac{z^{n+1}}{z^n} \right| = |z|.$$

Hence it converges for $|z| < 1$ and has $R = 1$.

Root test

For a series $\sum c_n$, suppose

$$L = \lim_{n \rightarrow \infty} |c_n|^{1/n}$$

exists. Again, $L < 1$ gives absolute convergence, $L > 1$ gives divergence, and $L = 1$ is inconclusive.

Example I: For $c_n = z^n$,

$$L = \lim_{n \rightarrow \infty} |z^n|^{1/n} = |z|,$$

which gives the same disk $|z| < 1$.

Example II: For $\sum_{n=0}^{\infty} \left(\frac{2n+3}{5n-4}\right)^n$

$$L = \lim_{n \rightarrow \infty} \left| \left(\frac{2n+3}{5n-4}\right)^n \right|^{1/n} = \lim_{n \rightarrow \infty} \left| \frac{2n+3}{5n-4} \right| = \frac{2}{5} < 1.$$

Therefore the series converges absolutely.

Radius directly from the coefficients

For the power series

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n,$$

the Cauchy–Hadamard formula gives

$$\frac{1}{R} = \limsup_{n \rightarrow \infty} |a_n|^{1/n}.$$

When the ratios are defined and the following limit exists, the ratio form is often faster:

$$R = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|.$$

Method choice

Use the root form (Cauchy–Hadamard formula) for general coefficients and the ratio form for factorials or products.

Worked example: ratio form with factorial coefficients

Find the radius of convergence of

$$\sum_{n=1}^{\infty} \frac{n!}{n^n} z^n.$$

The coefficients contain a factorial, so the ratio form is the natural choice:

$$\left| \frac{a_n}{a_{n+1}} \right| = \frac{n!}{n^n} \cdot \frac{(n+1)^{n+1}}{(n+1)!} = \left(1 + \frac{1}{n}\right)^n \rightarrow e.$$

$$\boxed{R = e}$$

Why the ratio form works here

The factorial $n!$ is canceled exactly between consecutive coefficients, leaving a standard limit $(1 + 1/n)^n \rightarrow e$.

Worked example: root form with an n -th power

Find the radius of convergence of

$$\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{n^2} z^n.$$

Here $a_n = \left(1 + \frac{1}{n}\right)^{n^2}$, and the ratio a_n/a_{n+1} is algebraically messy. The root form is direct:

$$|a_n|^{1/n} = \left(1 + \frac{1}{n}\right)^n \longrightarrow e, \quad \text{so} \quad \boxed{R = \frac{1}{e}}.$$

Boundary check

At $z = 1/e$, the terms satisfy $\left(1 + \frac{1}{n}\right)^{n^2} e^{-n} \rightarrow e^{-1/2} \neq 0$, so the series diverges on the whole boundary circle.

Worked example: missing coefficients and the limsup

Find the radius of convergence of

$$\sum_{n=1}^{\infty} n z^{2n}.$$

Equivalently $a_{2n} = n$ and $a_{2n+1} = 0$: many coefficients vanish, so the ratio form does not apply directly. Cauchy–Hadamard handles this without case analysis:

$$\limsup_{n \rightarrow \infty} |a_n|^{1/n} = \lim_{n \rightarrow \infty} n^{1/(2n)} = 1 \implies \boxed{R = 1}.$$

Two ways to see it

- Root/limsup view: only the even indices matter in the limsup.
- Substitution view: with $w = z^2$, the series is $\sum n w^n$, which has radius 1 in w .

Two extreme radii

Infinite radius:

Find the radius of convergence of

$$\sum_{n=0}^{\infty} \frac{z^n}{(n!)^2}.$$

The ratio form is chosen as

$$R = \lim_{n \rightarrow \infty} \left| \frac{c_n}{c_{n+1}} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)^2}{|z|} = \infty.$$

Hence

$$\boxed{R = \infty}.$$

Zero radius:

Find the radius of convergence of

$$\sum_{n=0}^{\infty} n! z^n.$$

The ratio form is chosen as

$$R = \lim_{n \rightarrow \infty} \left| \frac{c_n}{c_{n+1}} \right| = \lim_{n \rightarrow \infty} \frac{1}{(n+1)|z|} = 0$$

for $z \neq 0$. Hence

$$\boxed{R = 0}.$$

Power series may be operated on term by term

If

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n,$$

then, for every closed disk $|z - z_0| \leq r < R$, the convergence is uniform. Hence

$$\begin{aligned} f'(z) &= \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}, \\ \int_{z_0}^z f(\zeta) d\zeta &= \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z - z_0)^{n+1}. \end{aligned}$$

Practical consequence

Differentiation and integration preserve the radius of convergence.

Worked example: integrate a geometric series

Starting from

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \quad |z| < 1,$$

integrate from 0 to z :

$$-\operatorname{Log}(1-z) = \sum_{n=0}^{\infty} \frac{z^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{z^n}{n}, \quad |z| < 1.$$

The operation is valid because the path lies inside the convergence disk.

Boundary behavior must be checked point by point

Consider

$$\sum_{n=1}^{\infty} \frac{z^n}{n}, \quad R = 1.$$

On the boundary $|z| = 1$, different points behave differently:

$$z = 1 : \quad \sum_{n=1}^{\infty} \frac{1}{n}$$

diverges.

$$z = -1 : \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

converges by the alternating-series test.

Conclusion

The radius determines the open convergence disk, but it does not decide the boundary.

The exponential series is entire

The exponential function was defined as the series

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

For $c_n = z^n/n!$,

$$\left| \frac{c_{n+1}}{c_n} \right| = \frac{|z|}{n+1} \longrightarrow 0.$$

Thus e^z converges for every $z \in \mathbb{C}$ and $R = \infty$.

Term-by-term differentiation gives

$$\frac{d}{dz} e^z = \sum_{n=0}^{\infty} \frac{d}{dz} \frac{z^n}{n!} = \sum_{n=1}^{\infty} \frac{nz^{n-1}}{n!} = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z.$$

This proves the exponential property $\frac{d}{dz} e^z = e^z$.

A transformed geometric series

Find the convergence region of

$$\sum_{n=0}^{\infty} \left(\frac{z-2}{z+1} \right)^n.$$

The root test gives

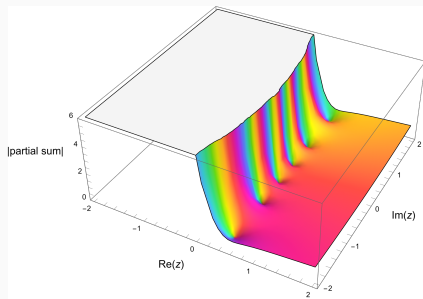
$$\left| \frac{z-2}{z+1} \right| < 1 \iff |z-2| < |z+1|.$$

For $z = x + iy$,

$$(x-2)^2 + y^2 < (x+1)^2 + y^2 \iff x > \frac{1}{2}.$$

Thus the series converges in the half-plane $\operatorname{Re} z > 1/2$.

Visualizing a truncated transformed series



The surface shows the modulus of

$$\sum_{k=0}^8 \left(\frac{z-2}{z+1} \right)^k.$$

The rapid growth occurs where the geometric ratio has modulus greater than 1.

A truncated approximation ($n = 8$).

See the Wolfram notebook for details.

Taylor series

From power series to analytic functions

Inside its radius of convergence, a power series defines an analytic function.

The converse question

Can every analytic function be represented by a power series?

Taylor's theorem answers yes and determines the coefficients from derivatives at the center. If the series is truncated, the result is a polynomial approximation.

Taylor's theorem

Theorem

If $f(z)$ is analytic in a region Ω containing z_0 , then:

- It can be represented by

$$\begin{aligned} f(z) &= f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots + \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n + \cdots \\ &= \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k \end{aligned}$$

The representation is called the **Taylor series** of f about z_0 .

- Taylor series is valid in the largest open disk $|z - z_0| < R$ contained in Ω ; i.e. the radius of convergence equals the radius of that largest disk.

The idea: approximate f by terms that are smaller and smaller; truncating the series gives a polynomial approximation.

Proof: recursive decomposition

Given $f(z)$, define

$$f_1(z) = \begin{cases} \frac{f(z) - f(z_0)}{z - z_0}, & z \neq z_0, \\ f'(z_0), & z = z_0, \end{cases}$$

analytic in the disk. Then

$$f(z) = f(z_0) + (z - z_0)f_1(z).$$

Repeat the construction:

$$f_{j-1}(z) = f_{j-1}(z_0) + (z - z_0)f_j(z).$$

After n steps,

$$f(z) = \sum_{j=0}^n (z - z_0)^j f_j(z_0) + (z - z_0)^{n+1} f_{n+1}(z).$$

The last term is the remainder.

Proof: identify the coefficients

Differentiating the finite decomposition at z_0 gives

$$f_j(z_0) = \frac{f^{(j)}(z_0)}{j!}.$$

Therefore

$$f(z) = \sum_{j=0}^n \frac{f^{(j)}(z_0)}{j!} (z - z_0)^j + R_n(z),$$

where

$$R_n(z) = (z - z_0)^{n+1} f_{n+1}(z).$$

It remains to prove that $R_n(z) \rightarrow 0$ as $n \rightarrow \infty$.

Proof: the remainder tends to zero

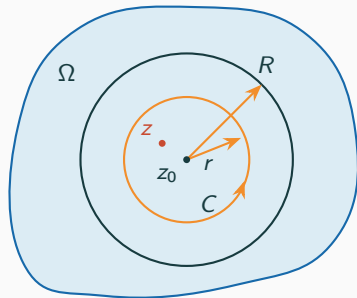
Cauchy's formula for derivatives, with a circle C of radius r with $|z - z_0| < r < R$, gives

$$f^{(n+1)}(z) = \frac{(n+1)!}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z)^{n+2}} d\zeta$$

and

$$\begin{aligned} |R_n(z)| &= |(z - z_0)^{n+1} f_{n+1}(z)| \\ &= \frac{1}{2\pi} |z - z_0|^{n+1} \left| \oint_C \frac{f(\zeta)}{(\zeta - z)^{n+2}} d\zeta \right| \\ &\leq \frac{1}{2\pi} |z - z_0|^{n+1} \cdot \frac{2\pi r \cdot \max_C |f|}{(2r)^{n+2}} \\ &= \frac{1}{2} \left(\frac{|z - z_0|}{2r} \right)^{n+1} \max_C |f| \end{aligned}$$

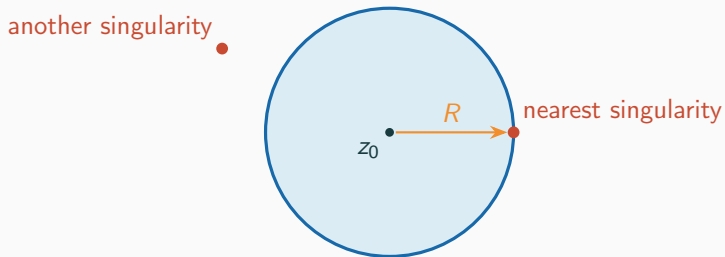
Since $|z - z_0|/r < 1$, we have $R_n(z) \rightarrow 0$.



Regions in the proof of Taylor's theorem.

Radius and the nearest singularity

The Taylor series about z_0 converges in the largest open disk on which f is analytic.



Thus R equals the distance from z_0 to the nearest singularity of f .

Taylor coefficients: an analytical workflow

For a function analytic near z_0 , determine the expansion systematically:

1. Differentiate to obtain $f^{(n)}(z)$.
2. Evaluate the derivative at the center, $f^{(n)}(z_0)$.
3. Divide by $n!$ to obtain the coefficient

$$a_n = \frac{f^{(n)}(z_0)}{n!}.$$

Hence the N th-order local model is

$$f(z) = \sum_{n=0}^N \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n + R_N(z).$$

Maclaurin coefficients of familiar functions

For $z_0 = 0$, Taylor's theorem reproduces

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!},$$

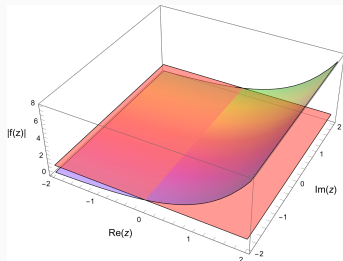
$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!},$$

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}.$$

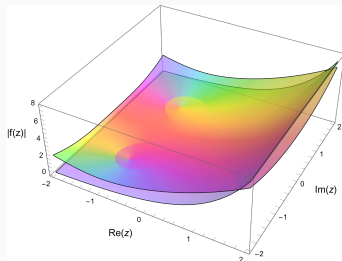
For the exponential function, $f^{(n)}(0) = 1$, so

$$a_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{n!}.$$

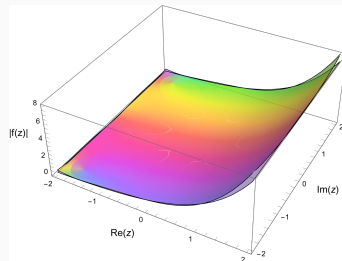
Taylor approximation of e^z



$n = 0$



$n = 2$



$n = 5$

$|e^z|$ and its Taylor polynomial about $z = 0$.

As n increases, the polynomial agrees with the function on a larger visible region.

See the Wolfram notebook for details.

Worked example: a rational function

Let

$$f(z) = \frac{z^2 - 2}{z - 1}.$$

Since $f(z) = z + 1 + 1/(1 - z)$, the Taylor series about 0 is

$$\boxed{f(z) = 2 + 2z + \sum_{n=2}^{\infty} z^n}, \quad |z| < 1.$$

The nearest singularity is the pole at $z = 1$, so $R = 1$.

Worked example: $\text{Log}(1 + z)$

Start from the geometric expansion

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n, \quad |z| < 1.$$

Integrating term by term from 0 to z gives

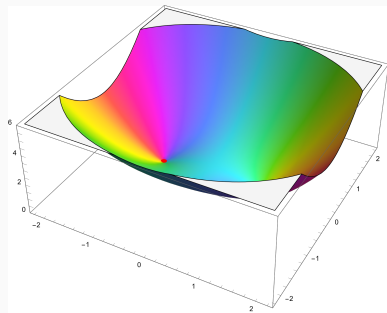
$$\boxed{\text{Log}(1+z) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n}, \quad |z| < 1.}$$

The nearest singularity is at $z = -1$, hence $R = 1$.

Isolated zeros

A zero z_0 of f is **isolated** if $f(z) \neq 0$ in some punctured neighborhood of z_0 .

For $f(z) = z^2 - 1$, the zeros $z = \pm 1$ are isolated.



Isolated zeros of $z^2 - 1$.

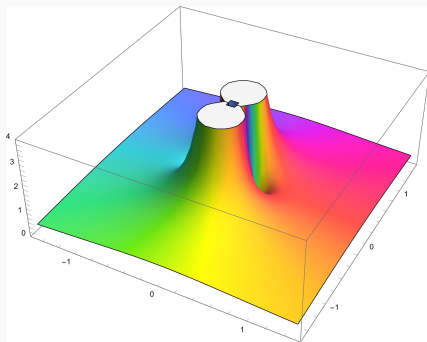
See the Wolfram notebook for details.

A non-isolated zero

Consider

$$f(z) = \begin{cases} \sin(1/z), & z \neq 0, \\ 0, & z = 0. \end{cases}$$

The zeros $z = 1/(k\pi)$ accumulate at 0, so the zero at 0 is not isolated.



Zeros of $\sin(1/z)$ accumulating at 0.

See the Wolfram notebook for details.

Zeros of analytic functions are isolated

Theorem

All zeros of an analytic function f are isolated, except when f is identically zero.

Proof. Let a_k be the first nonzero coefficient of the Taylor series at the zero z_0 . Then

$$f(z) = (z - z_0)^k (a_k + a_{k+1}(z - z_0) + \cdots) = (z - z_0)^k g(z),$$

where $g(z_0) = a_k \neq 0$. By continuity $g(z) \neq 0$ near z_0 , and $(z - z_0)^k \neq 0$ in a punctured neighborhood of z_0 . Hence z_0 is isolated. ■

Order of a zero

Let f be analytic, not identically zero, with a zero at z_0 . The **order** of the zero is the smallest h such that

$$f^{(h)}(z_0) \neq 0.$$

Equivalently, near z_0 ,

$$f(z) = (z - z_0)^h f_h(z), \quad f_h \text{ analytic, } f_h(z_0) \neq 0.$$

Reading the order from the series

The order is the index of the first nonzero Taylor coefficient at z_0 .

Worked examples: a simple zero and a double zero

Example I:

$$f(z) = 2(z - 1).$$

$f(1) = 0$ and $f'(1) = 2 \neq 0$: the zero at $z = 1$ has **order 1** (simple zero).

Example II:

$$f(z) = \frac{(z^2 + 1)^2}{z}.$$

Factor around $z = i$:

$$f(z) = (z - i)^2 \underbrace{\frac{(z + i)^2}{z}}_{g(z)}, \quad g(i) = 4i \neq 0.$$

The zero at $z = i$ has **order 2**.

Check: $f'(i) = 0$, $f''(i) = 8i \neq 0$.

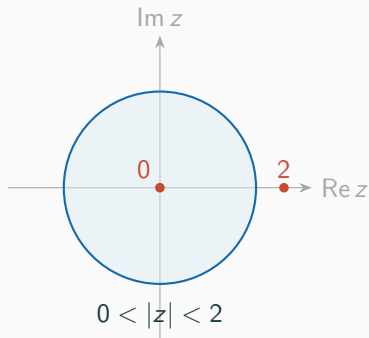
Laurent Series

Taylor series stop at a singular center

Consider the case of a rational function, such as

$$f(z) = \frac{z + 1}{z(z - 2)}.$$

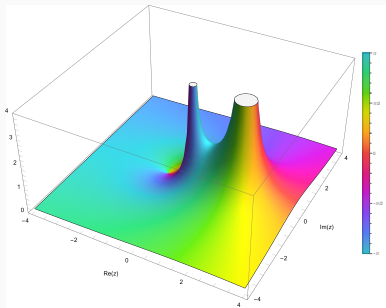
It is singular at $z = 0$ and $z = 2$, so a Taylor series centered at 0 cannot represent it.



Required extension

Allow negative powers of $z - z_0$. The puncture excludes the singular center, while an outer boundary records the next singularity.

The visible singularities



The function $f(z) = \frac{z+1}{z(z-2)}$.

For

$$f(z) = \frac{z+1}{z(z-2)},$$

the magnitude develops peaks at the two singular points 0 and 2.

The plot motivates separate Laurent expansions in $0 < |z| < 2$ and $|z| > 2$.

Different annular expansions of one function

Partial fractions give

$$\frac{z+1}{z(z-2)} = -\frac{1}{2z} + \frac{3}{2(z-2)}.$$

The geometric-series choice depends on the region:

$$0 < |z| < 2 : \quad f(z) = -\frac{1}{2z} - \frac{3}{2} \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}},$$

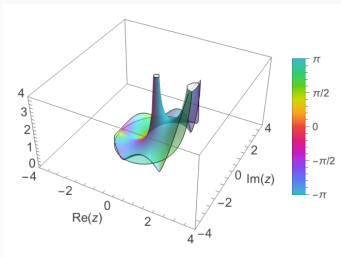
$$|z| > 2 : \quad f(z) = \frac{1}{z} + 3 \sum_{n=0}^{\infty} \frac{2^n}{z^{n+2}}.$$

Key idea

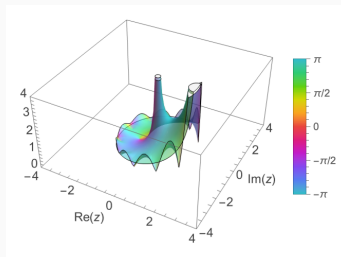
A Laurent expansion is inseparable from its annulus of convergence.

Successive approximations for $\frac{z+1}{z(z-2)}$ in $0 < |z| < 2$

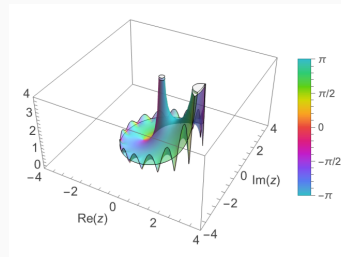
$$0 < |z| < 2: \quad f(z) = \frac{z+1}{z(z-2)} = -\frac{1}{2z} - \frac{3}{2} \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$



$n = 5$



$n = 10$



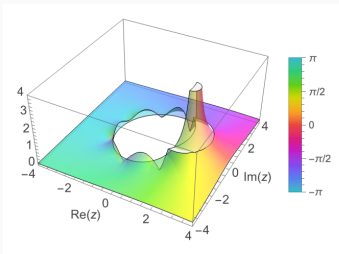
$n = 15$

Successive approximations by Laurent series in $0 < |z| < 2$.

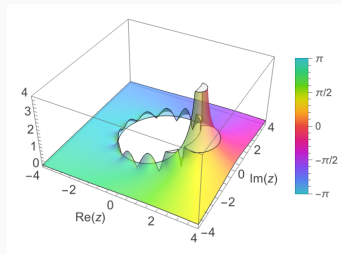
See the Wolfram notebook for details.

Successive approximations for $\frac{z+1}{z(z-2)}$ in $|z| > 2$

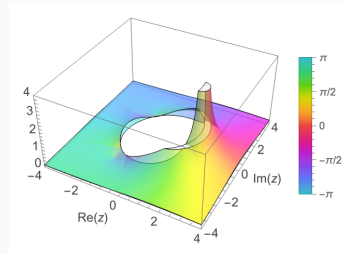
$$|z| > 2: \quad f(z) = \frac{1}{z} + 3 \sum_{n=0}^{\infty} \frac{2^n}{z^{n+2}}.$$



$n = 5$



$n = 10$



$n = 15$

Successive approximations by Laurent series in $|z| > 2$.

See the Wolfram notebook for details.

Definition of Laurent series

A Laurent series centered at z_0 is

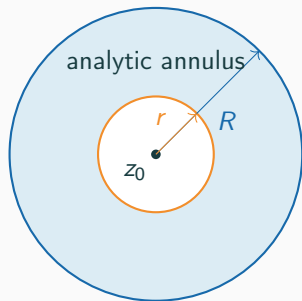
$$\boxed{\sum_{n=-\infty}^{\infty} c_n (z - z_0)^n} = \underbrace{\sum_{n=0}^{\infty} c_n (z - z_0)^n}_{\text{regular part}} + \underbrace{\sum_{n=1}^{\infty} \frac{c_{-n}}{(z - z_0)^n}}_{\text{principal part}}.$$

It converges in an annulus

$$r < |z - z_0| < R.$$

The inner radius is created by negative powers; the outer radius is limited by the next singularity.

An annulus replaces the Taylor disk



- Taylor: $|z - z_0| < R$.
- Laurent: $r < |z - z_0| < R$.
- Different annuli generally produce different coefficients.
- Always state the region with the expansion.

Theorem

If f is analytic in $r < |z - z_0| < R$, then it has a unique Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n,$$
$$c_n = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta,$$

where C is any positively oriented circle in the annulus.

Unlike Taylor coefficients, these coefficients need not be obtained from derivatives at the singular center.

Why Laurent's theorem works

Choose two circles surrounding z inside the annulus. Cauchy's formula gives an outer contribution minus an inner contribution.

On the outer contour, $|z - z_0| < |\zeta - z_0|$:

$$\frac{1}{\zeta - z} = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}.$$

On the inner contour, $|\zeta - z_0| < |z - z_0|$:

$$\frac{1}{\zeta - z} = - \sum_{n=0}^{\infty} \frac{(\zeta - z_0)^n}{(z - z_0)^{n+1}}.$$

The two geometric expansions generate the regular and principal parts.

Worked example: a simple pole inside a disk

Find the Laurent series expansion of $f(z) = (z + 1)/z(z - 1)$ around $z_0 = 0$ in the region $0 < |z| < 1$.

For

$$f(z) = \frac{z + 1}{z(z - 1)} = -\frac{1}{z} + \frac{2}{z - 1},$$

use $1/(z - 1) = -1/(1 - z)$ when $|z| < 1$:

$$\boxed{f(z) = -\frac{1}{z} - 2 \sum_{n=0}^{\infty} z^n}, \quad 0 < |z| < 1.$$

The principal part consists of one term, $-1/z$.

Worked example: expansion about a double pole

Find the Laurent series expansion of the function $f(z) = e^z/(z-1)^2$ around $z = 1$.

Let $w = z - 1$. Then

$$\frac{e^z}{(z-1)^2} = \frac{e e^w}{w^2} = \frac{e}{w^2} + \frac{e}{w} + e \sum_{n=0}^{\infty} \frac{w^n}{(n+2)!}.$$

Therefore

$$\boxed{\frac{e^z}{(z-1)^2} = \frac{e}{(z-1)^2} + \frac{e}{z-1} + e \sum_{n=0}^{\infty} \frac{(z-1)^n}{(n+2)!}}.$$

Two negative-power terms reveal a pole of order 2.

Worked example: the same function in two regions

Find the Laurent expansion of $f(z) = 1/z(z+1)$ around $z = 0$.

$f(z)$ can be written as $f(z) = 1/[z(z+1)] = z^{-1}(1+z)^{-1}$.

If $0 < |z| < 1$,

If $|z| > 1$,

$$f(z) = \sum_{n=0}^{\infty} (-1)^n z^{n-1}.$$

$$f(z) = \sum_{n=0}^{\infty} (-1)^n z^{-n-2}.$$

The expansion contains one negative power.

The expansion contains only negative powers.

Method

Factor the expression so that the geometric-series ratio has modulus less than one in the requested region.

Holomorphic and Meromorphic Functions

From holomorphic to singular functions

A function is **holomorphic** (analytic) in Ω if it is complex-differentiable at every point of Ω .

Entire functions

A function is **entire** if it is analytic for all $z \in \mathbb{C}$. Examples: e^z , $\sin z$, $\cos z$, and all polynomials.

Most functions are not entire. A point where f fails to be analytic is called a **singularity**. The rest of this section classifies the possible singularities of a non-entire function.

Classification of singularities

Isolated singularities

- Removable singularities
- Poles (of finite order)
- Essential isolated singularities

Non-isolated singularities

- Branch points
- Points of accumulation of other singularities

Isolated vs. non-isolated

A singularity z_0 is **isolated** if there exists a punctured neighborhood $0 < |z - z_0| < \epsilon$ in which f is analytic. Otherwise it is **non-isolated**.

Definition of removable singularities

Definition

$f(z)$ has a **removable singularity** at z_0 if f can be made analytic at z_0 by redefining its value there.

Examples:

- $f(z) = \frac{\sin z}{z}$ at $z_0 = 0$; redefine $f(0) = 1$.
- $f(z) = \begin{cases} 0, & z \neq 0 \\ 1, & z = 0 \end{cases}$ at $z_0 = 0$; redefine $f(0) = 0$.

In both cases a single value repair removes the singularity.

Two criteria for removable singularities

1st criterion (boundedness)

If f is analytic and bounded by the same M in every punctured disk $0 < |z - z_0| < \epsilon$, then z_0 is removable.

2nd criterion (limit)

If f is analytic in $\Omega \setminus \{z_0\}$, then z_0 is removable if and only if

$$\lim_{z \rightarrow z_0} (z - z_0) f(z) = 0.$$

When the 2nd criterion holds, the value that repairs f is

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z_0} d\zeta.$$

Both proofs follow the pattern of Cauchy's integral formula.

Worked example: $\frac{\sin z}{z}$ at 0

Apply the 2nd criterion to $f(z) = \frac{\sin z}{z}$:

$$\lim_{z \rightarrow 0} z \cdot \frac{\sin z}{z} = \lim_{z \rightarrow 0} \sin z = 0.$$

Therefore $z = 0$ is a removable singularity. The repaired value is

$$f(0) = \lim_{z \rightarrow 0} \frac{\sin z}{z} = 1.$$

Equivalently, the Taylor expansion

$$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots$$

has **no negative powers**, confirming that the principal part is empty.

First definition of Poles (order via limits)

Definition

$f(z)$ has a **pole of order** h at z_0 if:

1. z_0 is an isolated singularity of f , and
2. there is a positive integer h such that

$$\lim_{z \rightarrow z_0} |z - z_0|^\alpha |f(z)| = \begin{cases} 0, & \alpha > h, \\ \infty, & \alpha < h. \end{cases}$$

A pole of order 1 is a **simple pole**; order 2 is a **double pole**, etc. Locally, $f(z) \sim b_h(z - z_0)^{-h}$: the negative power dominates.

Second definition of Poles (via Laurent series)

Definition

f has a pole at z_0 if, in $0 < |z - z_0| < \epsilon$, its Laurent expansion

$$f(z) = \sum_{n=0}^{\infty} c_n(z - z_0)^n + \sum_{n=1}^{\infty} \frac{c_{-n}}{(z - z_0)^n}$$

has at least one but only **finitely many** nonzero b_n . The largest index h with $b_h \neq 0$ is the **order**.

Equivalence of the two definitions. If z_0 is a pole of order h in the first sense, then $(z - z_0)^h f(z)$ has a removable singularity and admits a Taylor series

$$(z - z_0)^h f(z) = \sum_{n=0}^{\infty} B_n(z - z_0)^n.$$

Dividing by $(z - z_0)^h$ recovers the Laurent form with highest negative power $-h$. The converse is immediate.

Worked example: poles of a rational function

Let

$$f(z) = \frac{z^2 - 1}{z(z - 2)^3(z - i)}.$$

The denominator zeros give candidate poles at $z = 0, 2, i$. Reading the factor multiplicities:

Pole	Factor in denominator	Order
$z = 0$	z^1	1 (simple)
$z = 2$	$(z - 2)^3$	3 (triple)
$z = i$	$(z - i)^1$	1 (simple)

The Laurent expansion about $z = 2$ begins

$$f(z) = \frac{\cdots}{(z - 2)^3} + \frac{\cdots}{(z - 2)^2} + \frac{\cdots}{z - 2} + \cdots,$$

so the highest negative power is $(z - 2)^{-3}$, confirming order 3.

Definition

A function is **meromorphic** in a region Ω if it is analytic there except at isolated poles.

- The sum, product, and quotient of meromorphic functions are meromorphic (provided no denominator vanishes identically).
- In particular, every rational function is meromorphic on all of $\mathbb{C}$.
- **Theorem:** every function meromorphic on all of $\mathbb{C}$ is the quotient of two entire functions.

Meromorphic functions are the natural bridge from holomorphic functions to the residue calculus of L06: only the $(z - z_0)^{-1}$ coefficient will matter for contour integration.

Worked example: $\frac{\sin z}{z^3}$ at 0

Expand the numerator:

$$\frac{\sin z}{z^3} = \frac{1}{z^3} \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \right) = \frac{1}{z^2} - \frac{1}{6} + \frac{z^2}{120} - \cdots$$

The highest negative power is z^{-2} , so $z = 0$ is a **pole of order 2**. Cross-check with the limit definition:

$$\lim_{z \rightarrow 0} z^3 \cdot \frac{\sin z}{z^3} = 0, \quad \lim_{z \rightarrow 0} |z| \cdot \left| \frac{\sin z}{z^3} \right| = \infty,$$

which is consistent with order $h = 2$ (the cutoff lies between $\alpha = 1$ and $\alpha = 3$).

Definition

$f(z)$ has a **pole of order n at infinity** if:

1. f is analytic in $|z| > R$ for some R (a neighborhood of ∞), and
2. $g(z) = f(1/z)$ has a pole of order n at $z = 0$.

The change of variable $w = 1/z$ maps a neighborhood of ∞ to a punctured neighborhood of 0, reducing the question to the finite-plane theory.

Worked example: a pole at infinity

Does $f(z) = \frac{z^2}{z-2}$ have a pole at ∞ , and of what order?

Set $g(z) = f(1/z)$:

$$g(z) = \frac{(1/z)^2}{(1/z) - 2} = \frac{1}{z(1-2z)} = \frac{1}{z} \cdot \frac{1}{1-2z}.$$

Using the geometric series for $|2z| < 1$:

$$g(z) = \frac{1}{z} (1 + 2z + 4z^2 + 8z^3 + \cdots) = \frac{1}{z} + 2 + 4z + 8z^2 + \cdots$$

The highest negative power is z^{-1} , so g has a simple pole at 0.

$f(z)$ has a simple pole at ∞ .

Worked example: e^z at infinity is not a pole

For $f(z) = e^z$, set $g(z) = f(1/z) = e^{1/z}$. Its Laurent expansion is

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots,$$

which has **infinitely many** negative powers. Hence g does not have a pole at 0.

Equivalently, for every integer n ,

$$\lim_{z \rightarrow 0} z^n e^{1/z}$$

does not exist (it is indeterminate), so no finite order h satisfies the pole definition.

e^z has an essential singularity at ∞ .

Essential singularities: two definitions

1st definition (limits)

z_0 is an essential singularity if it is isolated but there is **no** integer h such that

$$|z - z_0|^\alpha |f(z)| \rightarrow \begin{cases} 0, & \alpha > h \\ \infty, & \alpha < h \end{cases}$$

i.e. the pole-order test fails for every h .

2nd definition (Laurent)

z_0 is essential if the Laurent principal part

$$\sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

contains **infinitely many** nonzero terms.

The two definitions are equivalent, just as for poles.

Worked example: infinitely many negative powers

Find the Laurent series of $f(z) = e^{1/z}$ around $z = 0$.

The exponential series is

$$e^z = 1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}z^n.$$

Substitute $1/z$ into the exponential series:

$$\boxed{e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots} = \sum_{n=0}^{\infty} \frac{1}{n!z^n}, \quad z \neq 0.$$

The principal part has infinitely many nonzero terms, so $z = 0$ is an **essential singularity**. No power $(z - 0)^h$ can tame the singularity: for every h , $z^h e^{1/z}$ still contains negative powers after multiplication.

Theorem (Casorati–Weierstrass)

In every neighborhood of an essential singularity, $f(z)$ comes arbitrarily close to **any** prescribed complex value.

In other words, the image of every punctured disk $0 < |z - z_0| < \epsilon$ under f is dense in $\mathbb{C}$.

This is dramatically different from poles (where $|f| \rightarrow \infty$) and removable singularities (where f has a finite limit). It explains why essential singularities cannot be removed or assigned a finite/infinite order.

The Laurent principal part classifies an isolated singularity

Suppose f is analytic in $0 < |z - z_0| < R$.

Laurent principal part	Type at z_0	Local behavior
no negative powers	removable	finite analytic extension
finitely many	pole of order m	$(z - z_0)^{-m}$ dominates
infinitely many	essential	no finite pole order

Reading rule: Expand first; then inspect the negative-power terms.

$$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots \quad \Rightarrow \text{removable at } 0,$$

$$\frac{\sin z}{z^3} = \frac{1}{z^2} - \frac{1}{3!} + \frac{z^2}{5!} - \dots \quad \Rightarrow \text{pole of order } 2,$$

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \dots \quad \Rightarrow \text{essential at } 0.$$

Branch points

A **branch point** is a singularity around which a function cannot be defined single-valuedly; a branch cut (a discontinuity) must emanate from it. Because the cut extends to the boundary, the singularity is **not isolated**.

Example: $\text{Log } z$ at 0

The principal logarithm has a branch cut conventionally placed along the negative real axis. The discontinuity departs from $z = 0$, so 0 is a branch point.

Example: $\sqrt{(z-1)(z+1)}$

Branch points at $z = \pm 1$; a branch cut connects them (or runs to infinity). The function jumps across the cut.

Points of accumulation of singularities

A point z_0 is a **point of accumulation of singularities** if every punctured neighborhood of z_0 contains infinitely many singularities of f .

- $f(z) = \frac{1}{\sin(\pi/z)}$: singularities at $z = 1/n$ accumulate at 0.
- The geometric series $\sum_{n=0}^{\infty} z^n = 1/(1-z)$: every point on the unit circle $|z| = 1$ is a singularity of the analytic continuation, so none of them is isolated.

At such a point, no punctured disk is free of singularities, so the isolated-singularity classification (removable / pole / essential) does not apply.

What we have learned

- A convergent series has convergent partial sums and terms tending to zero.
- Root and ratio methods determine R ; boundary points require separate tests.
- Inside its convergence disk, a power series supports termwise operations.
- Taylor series represent analytic functions by local derivative data.
- Zeros are described by the first nonzero Taylor coefficient.
- Laurent series represent functions in annuli and may include negative powers.
- The principal part classifies isolated singularities.

Next $\gg$ Lecture 06 — Residues