

L06 — Residues

Engineering Mathematics I

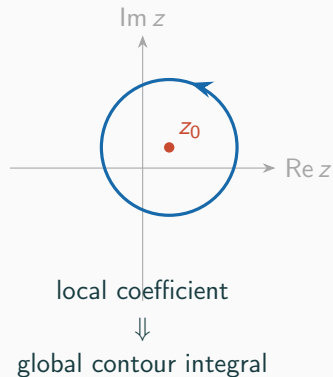
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Road map

- Isolated Singularities and Laurent Coefficients
- Residues and Their Computation
- The Residue Theorem
- Applications of Residues



From local singularities to contour integrals

L05 showed that near an isolated singularity z_0 ,

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n.$$

Most coefficients disappear when integrated around a closed contour:

$$\oint_C (z - z_0)^n dz = \begin{cases} 2\pi i, & n = -1, \\ 0, & n \neq -1. \end{cases}$$

Central idea

A contour integral extracts one local coefficient: the residue.

Isolated Singularities and Laurent Coefficients

A minimal review of isolated singularities

Let f be analytic in a punctured disk $0 < |z - z_0| < R$.

Removable

No negative powers:

$$c_{-1} = c_{-2} = \cdots = 0.$$

Pole of order m

A finite principal part:

$$c_{-m} \neq 0.$$

Essential

Infinitely many negative-power terms.

Only the coefficient c_{-1} will control the contour integral.

Definition of residue

If

$$f(z) = \cdots + \frac{c_{-2}}{(z - z_0)^2} + \frac{c_{-1}}{z - z_0} + c_0 + c_1(z - z_0) + \cdots ,$$

then the **residue of f at z_0** is

$$\boxed{\operatorname{Res}(f, z_0) = c_{-1}} .$$

- The residue is a number, not the whole principal part.
- It may be zero even when the singularity is not removable.
- It depends only on the local behavior near z_0 .

Why the coefficient c_{-1} is special

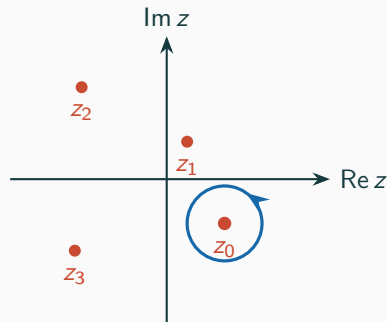
Parameterize C by $z = z_0 + re^{it}$, $0 \leq t \leq 2\pi$. Then

$$\oint_C (z - z_0)^n dz = ir^{n+1} \int_0^{2\pi} e^{i(n+1)t} dt \\ = 0, \quad n \neq -1,$$

whereas

$$\oint_C \frac{dz}{z - z_0} = 2\pi i.$$

Hence $\oint_C f(z) dz = 2\pi i c_{-1}$.



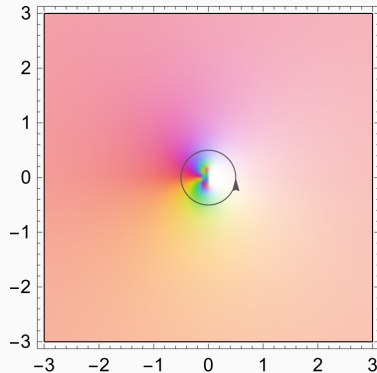
The residue at z_0 is related to the contour integral around it.

Residue as local circulation

For every sufficiently small positively oriented circle around z_0 ,

$$\operatorname{Res}(f, z_0) = \frac{1}{2\pi i} \oint_C f(z) \, dz .$$

The contour may be deformed inside the punctured analytic region without changing the value.



$f(z) = e^{1/z}$ and a small contour about 0.

See the Wolfram notebook for details.

Residues across the three singularity types

Singularity at z_0	Local form	Residue
Removable	$a_0 + a_1(z - z_0) + \dots$	0
Simple pole	$a_{-1}/(z - z_0) + \dots$	a_{-1}
Pole of order m	$\sum_{n=-m}^{\infty} a_n(z - z_0)^n$	a_{-1}
Essential	$\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$	a_{-1}

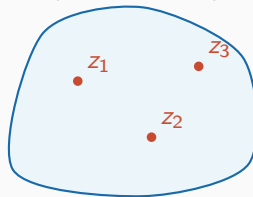
Do not confuse

The order of a pole is determined by the most negative power; the residue is always the coefficient of power -1 .

Why meromorphic functions are ideal for residues

A function is **meromorphic** in a region if it is analytic there except at isolated poles.

- Rational functions are meromorphic.
- Familiar analytic functions often form meromorphic quotients.
- Each isolated pole contributes one residue to a closed-contour integral.



analytic except at isolated poles

Zeros and poles are two sides of the same local factor

If an analytic function q has a zero of order m at z_0 , then

$$q(z) = (z - z_0)^m q_m(z), \quad q_m(z_0) \neq 0.$$

Hence, whenever $p(z_0) \neq 0$,

$$\frac{p(z)}{q(z)} = \frac{p(z)}{q_m(z)} \frac{1}{(z - z_0)^m}$$

has a pole of order m at z_0 .

Practical consequence

Before selecting a residue formula, compare the zero orders of the numerator and denominator and cancel common factors.

Residues and Their Computation

Four practical routes to a residue

1. Read the coefficient from a known Laurent expansion.
2. Use a limit at a simple pole.
3. Use a quotient shortcut when the denominator has a simple zero.
4. Use derivatives at a higher-order pole.

Method choice

Classify the singularity first. Do not compute a complete Laurent series when a local limit or derivative gives the coefficient directly.

Method I: read the Laurent coefficient

For

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots,$$

the coefficient of z^{-1} is 1. Therefore

$$\boxed{\operatorname{Res}(e^{1/z}, 0) = 1}.$$

Although $z = 0$ is an essential singularity, the residue is still obtained from one coefficient.

The Laurent expansion must match the contour

A function may have different Laurent expansions in different annuli. To integrate term by term around C , use an expansion whose annulus contains the entire contour.

Inner expansion

Valid for $0 < |z - z_0| < R$ and appropriate for a small contour.

Outer expansion

Valid for $|z - z_0| > R$ and appropriate only for a sufficiently large contour.

Critical check

The coefficient of $(z - z_0)^{-1}$ is meaningful only after the correct region of validity has been chosen.

worked example: correct annulus, correct residue

For $f(z) = 1/(z^3 - z^4)$ and $0 < |z| < 1$,

$$f(z) = \frac{1}{z^3} \frac{1}{1-z} = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + z + \cdots.$$

Thus $\text{Res}(f, 0) = 1$, and clockwise integration around $|z| = 1/2$ gives

$$\oint_C f(z) dz = -2\pi i.$$

The outer expansion

$$f(z) = -\frac{1}{z^4} - \frac{1}{z^5} - \cdots, \quad |z| > 1,$$

has no z^{-1} term but is invalid on this contour.

Series products can isolate the needed term

Consider $f(z) = e^z/z^3$. Since

$$\frac{e^z}{z^3} = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{2z} + \frac{1}{3!} + \cdots,$$

we obtain

$$\boxed{\operatorname{Res}\left(\frac{e^z}{z^3}, 0\right) = \frac{1}{2}}.$$

Only terms capable of producing power z^{-1} need to be retained.

Method II: the simple-pole limit

If f has a simple pole at z_0 , write

$$f(z) = \frac{c_{-1}}{z - z_0} + c_0 + c_1(z - z_0) + \cdots .$$

Then

$$\boxed{\operatorname{Res}(f, z_0) = \lim_{z \rightarrow z_0} (z - z_0)f(z)} .$$

Condition

This formula is valid for a simple pole. For a higher-order pole, the limit generally diverges.

Worked example: a simple pole

Let $f(z) = e^z/z$. The denominator has a simple zero at 0 and the numerator is nonzero there. Thus

$$\operatorname{Res}(f, 0) = \lim_{z \rightarrow 0} z \frac{e^z}{z} = e^0 = 1.$$

Hence every small positively oriented contour about 0 satisfies

$$\oint_C \frac{e^z}{z} dz = 2\pi i.$$

Method III: a quotient shortcut

Suppose $f = g/h$, where g and h are analytic near z_0 , with

$$h(z_0) = 0, \quad h'(z_0) \neq 0, \quad g(z_0) \neq 0.$$

Then z_0 is a simple pole and

$$\boxed{\operatorname{Res}\left(\frac{g}{h}, z_0\right) = \frac{g(z_0)}{h'(z_0)}}.$$

This is efficient for many rational and trigonometric quotients.

Two local residues of one function

For

$$f(z) = \frac{z^3 - 2}{z(z - 1)},$$

the simple poles are 0 and 1:

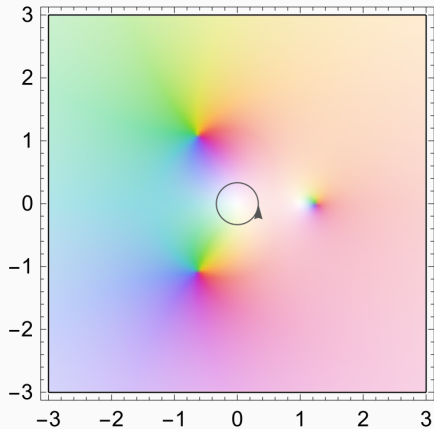
$$\operatorname{Res}(f, 0) = \left. \frac{z^3 - 2}{z - 1} \right|_{z=0} = 2,$$

$$\operatorname{Res}(f, 1) = \left. \frac{z^3 - 2}{z} \right|_{z=1} = -1.$$

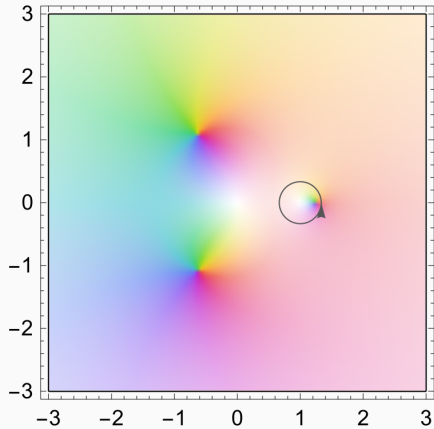
Locality

The same function has different residues at different singularities. The contour determines which local contribution is selected.

The contour selects the local contribution



Around $z = 0$: residue 2.



Around $z = 1$: residue -1 .

The phase pattern is global, but each small loop measures one local coefficient.

Method IV: a pole of order k

Suppose f has a pole of order k at z_0 . Its Laurent expansion is

$$f(z) = \frac{c_{-k}}{(z - z_0)^k} + \cdots + \frac{c_{-1}}{z - z_0} + c_0 + a_1(z - z_0) + c_2(z - z_0)^2 + \cdots$$

Multiplying by $(z - z_0)^k$ removes the singularity. The function

$$g(z) = (z - z_0)^k f(z) = c_{-k} + \cdots + b_{-1}(z - z_0)^{k-1} + c_0(z - z_0)^k + c_2(z - z_0)^{k+1} + \cdots$$

is analytic at z_0 (a removable singularity). The residue b_1 is precisely the coefficient of $(z - z_0)^{k-1}$ in the Taylor series of g :

$$b_1 = \frac{g^{(k-1)}(z_0)}{(k-1)!}$$

Substituting $g(z) = (z - z_0)^k f(z)$ yields the general formula

$$\boxed{\operatorname{Res}(f, z_0) = \frac{1}{(k-1)!} \left. \frac{d^{k-1}}{dz^{k-1}} [(z - z_0)^k f(z)] \right|_{z=z_0}}.$$

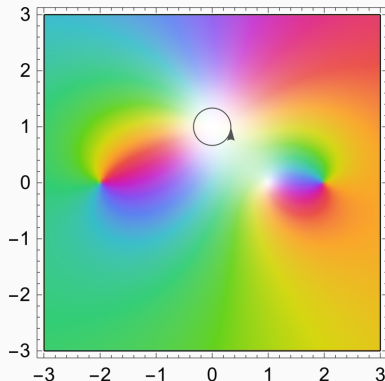
Worked example: a double pole

$$f(z) = \frac{z^2 - 4}{(z - 1)(z - i)^2}$$

$z = i$ is a double pole. Therefore

$$\begin{aligned}\operatorname{Res}(f, i) &= \left. \frac{d}{dz} \left(\frac{z^2 - 4}{z - 1} \right) \right|_{z=i} \\ &= \left. \frac{z^2 - 2z + 4}{(z - 1)^2} \right|_{z=i} = 1 + \frac{3}{2}i.\end{aligned}$$

- The small loop is centered at $z = i$.
- The local behavior is dominated by $(z - i)^{-2}$.
- The residue still comes from the subleading $(z - i)^{-1}$ term.



Worked example: simplify the apparent order

Compute the residue at $z = 1$ of

$$f(z) = \frac{\cos(\pi z/2)}{(z-1)^4}.$$

At $z = 1$, the numerator has a simple zero:

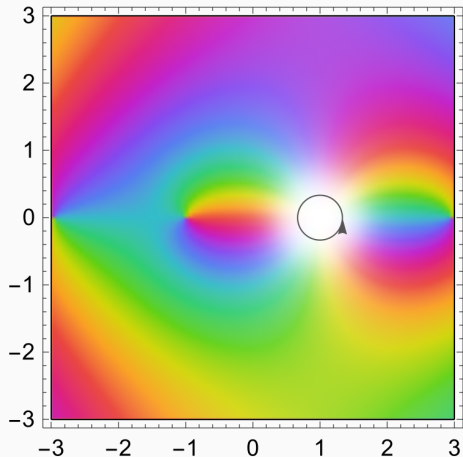
$$\cos \frac{\pi z}{2} = -\frac{\pi}{2}(z-1) + \frac{\pi^3}{48}(z-1)^3 + \dots$$

Thus the pole is of order 3, not 4, and

$$f(z) = -\frac{\pi}{2}(z-1)^{-3} + \frac{\pi^3}{48}(z-1)^{-1} + \dots$$

Therefore $\boxed{\operatorname{Res}(f, 1) = \pi^3/48}$.

Always simplify before applying a pole formula



$$f(z) = \frac{\cos(\pi z/2)}{(z-1)^4}$$

$$\text{Res}(f, 1) = \pi^3/48$$

Diagnostic rule

A denominator factor $(z - z_0)^m$ does not by itself prove a pole of order m . Check whether the numerator also vanishes.

Here one cancellation lowers the order from four to three.

Choosing the most efficient method

Local structure	Preferred tool	Typical situation
Known expansion	Read a_{-1}	Exponential or trigonometric series
Simple pole $g/h, h'(z_0) \neq 0$	One limit $g(z_0)/h'(z_0)$	Factored rational function Zero of a denominator
Pole of order m	$(m - 1)$ derivatives	Repeated denominator factor

Efficiency principle

Compute only the information required for the coefficient of $(z - z_0)^{-1}$.

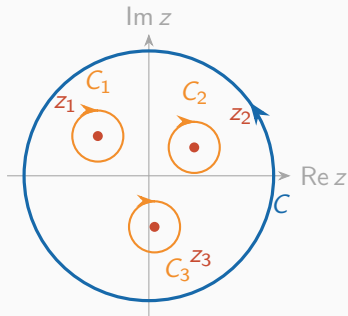
The Residue Theorem

From one singularity to many

Suppose a positively oriented contour C encloses isolated singularities $z_1, \dots, z_N$. Choose disjoint small circles C_k around them. Path deformation gives

$$\oint_C f(z) dz = \sum_{k=1}^N \oint_{C_k} f(z) dz.$$

Each local integral extracts one residue, so the global integral is obtained by adding local data.



$$\oint_C f(z) dz = \oint_{C_1} f(z) dz + \oint_{C_2} f(z) dz + \oint_{C_3} f(z) dz$$

Cauchy's residue theorem

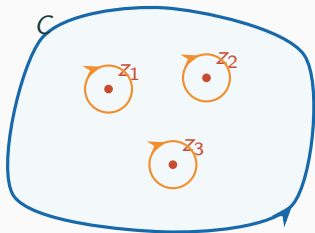
Theorem

Let C be a positively oriented simple closed contour. If f is analytic on C and inside C except at finitely many isolated singularities $z_1, \dots, z_N$, then

$$\oint_C f(z) \, dz = 2\pi i \sum_{k=1}^N \operatorname{Res}(f, z_k).$$

Singularities on the contour are not allowed: the integral then requires a different interpretation.

Geometric proof idea



Remove small disks around the singularities. The remaining region contains no singularity, so Cauchy's theorem applies to its complete boundary.

The inner circles inherit clockwise orientation; moving them to the other side produces the sum of positive local integrals.

$$\oint_C f(z) dz = \sum_{k=1}^N \oint_{C_k} f(z) dz = 2\pi i \sum_{k=1}^N \text{Res}(f, z_k).$$

Orientation controls the sign

For a positively oriented contour,

$$\oint_C f(z) dz = 2\pi i \sum_{k=1}^N \text{Res}(f, z_k).$$

Reversing the orientation changes the sign.



counterclockwise: positive



clockwise: negative

Worked example: one enclosed pole

Evaluate

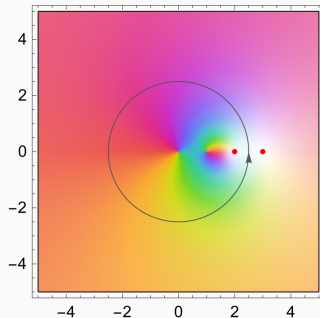
$$\oint_{|z|=5/2} \frac{z(z-1)}{(z-2)(z-3)} dz.$$

Only $z = 2$ lies inside the contour. Its residue is

$$\text{Res}(f, 2) = \left. \frac{z(z-1)}{z-3} \right|_{z=2} = -2.$$

Therefore

$$\oint_{|z|=5/2} f(z) dz = -4\pi i.$$



- Red points mark the poles 2 and 3.
- The circle $|z| = 5/2$ encloses only 2.
- The pole at 3 has no contribution to this integral.

Worked example: a rectangular contour

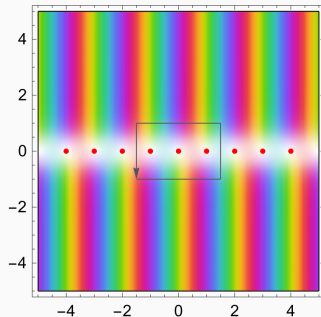
For $f(z) = 1/\sin(\pi z)$, every integer is a simple pole and

$$\operatorname{Res}\left(\frac{1}{\sin \pi z}, n\right) = \frac{1}{\pi \cos(\pi n)} = \frac{(-1)^n}{\pi}.$$

The rectangle with $-3/2 < \operatorname{Re} z < 3/2$ encloses $-1, 0, 1$, so

$$\sum \operatorname{Res} = -\frac{1}{\pi} + \frac{1}{\pi} - \frac{1}{\pi} = -\frac{1}{\pi}.$$

Hence $\oint_C dz/\sin(\pi z) = -2i$.

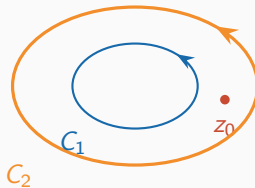


- The red points are integer poles.
- Only $-1, 0, 1$ lie inside the rectangle.
- Alternating residues make cancellation visible before calculation.

When a contour moves across a pole

Let C_1 and C_2 be two contours, with exactly one pole z_0 in the region between them. Then

$$\oint_{C_2} f(z) dz - \oint_{C_1} f(z) dz = 2\pi i \operatorname{Res}(f, z_0).$$



A practical contour checklist

Before applying the residue theorem, verify:

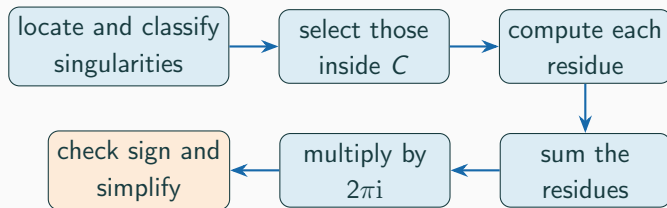
- Is the contour closed and piecewise smooth?
- What is its orientation?
- Is f analytic on the contour?
- Which isolated singularities are strictly inside?
- Which residue formula is most efficient at each one?

Critical warning

A pole on the contour violates the standard theorem. A principal-value construction is a separate topic.

Applications of Residues

The residue workflow



The difficult-looking contour integral becomes a finite local computation.

Worked example: mixed pole orders

Evaluate

$$\oint_{|z|=3} \frac{z+1}{z^2(z-2)} dz.$$

At the double pole $z = 0$,

$$\operatorname{Res}(f, 0) = \left. \frac{d}{dz} \frac{z+1}{z-2} \right|_{z=0} = -\frac{3}{4}.$$

At the simple pole $z = 2$, $\operatorname{Res}(f, 2) = 3/4$. The residues cancel, hence

$$\boxed{\oint_{|z|=3} f(z) dz = 0}.$$

A zero integral need not mean an analytic integrand

In the preceding example, f has two poles inside C , but

$$\operatorname{Res}(f, 0) + \operatorname{Res}(f, 2) = 0.$$

Therefore the contour integral vanishes by cancellation, not by Cauchy's theorem.

Diagnostic distinction

- Cauchy's theorem: no singularity inside the contour.
- Residue cancellation: singularities exist, but their total residue is zero.

Analytic behavior at infinity

On the extended plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, study f near infinity through

$$g(w) = f\left(\frac{1}{w}\right), \quad w \rightarrow 0.$$

Behavior of g at $w = 0$	Behavior of f at $z = \infty$
analytic or removable	analytic or removable at infinity
pole of order m	pole of order m at infinity
essential singularity	essential singularity at infinity

This completes the connection from the Riemann sphere in L01 to residue calculus.

Residue at infinity

If f has an isolated singularity at ∞ , its residue at infinity is

$$\operatorname{Res}(f, \infty) = -\frac{1}{2\pi i} \oint_C f(z) dz$$

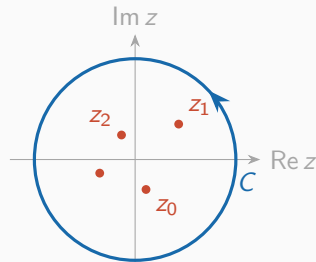
where C is a positively oriented contour enclosing all finite singularities.

Why the minus sign?

A contour that keeps ∞ on its left must be traversed clockwise. To retain the standard counterclockwise convention, the definition acquires an overall minus sign.

Under the change of variable $w = 1/z$, this is equivalent to

$$\operatorname{Res}(f, \infty) = -\operatorname{Res}\left(\frac{1}{w^2} f\left(\frac{1}{w}\right), 0\right).$$



all finite singularities inside C

Worked examples: residue at infinity

Example I

$$f(z) = \frac{3z^3 + 1}{z} = 3z^2 + \frac{1}{z}$$

In the expansion valid for large $|z|$, the coefficient of z^{-1} is 1. Therefore

$$\boxed{\operatorname{Res}(f, \infty) = -1}.$$

Example II

$$f(z) = \frac{z^5 + 3z^3 + 2}{2z(z+1)}$$

Polynomial division for large $|z|$ yields

$$f(z) = \frac{z^3}{2} - \frac{z^2}{2} + 2z - 2 + \frac{2}{z} + O\left(\frac{1}{z^2}\right).$$

The coefficient of z^{-1} is 2, hence

$$\boxed{\operatorname{Res}(f, \infty) = -2}.$$

Two equivalent routes

Read the coefficient of z^{-1} in the large- $|z|$ expansion and negate it. Equivalently, use the substitution formula $\operatorname{Res}(f, \infty) = -\operatorname{Res}(w^{-2}f(1/w), 0)$.

Residue theorem at infinity

If f is meromorphic on the extended complex plane, then

$$\sum_{z_k \in \mathbb{C}} \operatorname{Res}(f, z_k) + \operatorname{Res}(f, \infty) = 0.$$

Consequently,

$$\operatorname{Res}(f, \infty) = - \sum_{z_k \in \mathbb{C}} \operatorname{Res}(f, z_k).$$

If a positively oriented contour encloses all finite poles, then

$$\oint_C f(z) \, dz = -2\pi i \operatorname{Res}(f, \infty).$$

Computational advantage

When many finite poles lie inside a large contour, one residue at infinity may replace several local calculations.

Worked example: Residue theorem at infinity

For

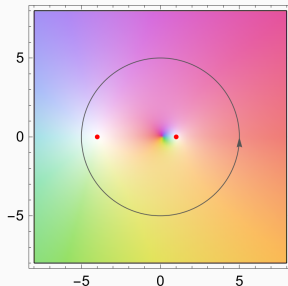
$$f(z) = \frac{8z - 1}{(z - 1)(z + 4)},$$

the large- z expansion begins with $8/z$, so $\text{Res}(f, \infty) = -8$. A positively oriented contour enclosing both finite poles gives

$$\oint_C f(z) dz = -2\pi i \text{Res}(f, \infty) = \boxed{16\pi i}.$$

Directly, the finite residues are $7/5$ and $33/5$, whose sum is 8 .

- Both poles -4 and 1 lie inside $|z| = 5$.
- Their finite residues sum to 8 .
- The same information is encoded by $\text{Res}(f, \infty) = -8$.



When is the infinity shortcut useful?

For a rational function $f = P/Q$:

- Expand $f(z)$ for large $|z|$ and read the coefficient of $1/z$.
- If $\deg Q \geq \deg P + 2$, then $\text{Res}(f, \infty) = 0$.
- If many finite poles are enclosed, one calculation at infinity may be faster.
- If only one or two finite poles are enclosed, direct local residues are usually clearer.

Method choice

Use infinity to simplify the calculation, not to hide which singularities control the integral.

Comprehensive example

Evaluate

$$\oint_{|z|=3} \frac{e^z}{(z-1)^2(z+2)} dz.$$

At $z = 1$,

$$\text{Res}(f, 1) = \left. \frac{d}{dz} \frac{e^z}{z+2} \right|_{z=1} = \frac{2e}{9}.$$

At $z = -2$, $\text{Res}(f, -2) = e^{-2}/9$. Therefore

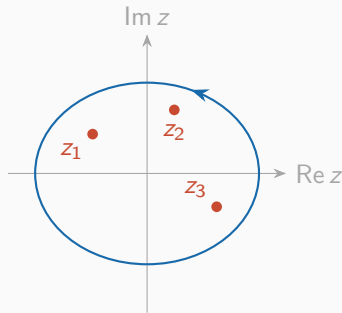
$$\boxed{\oint_{|z|=3} f(z) dz = \frac{2\pi i}{9} (2e + e^{-2})}.$$

Interpretation: local data build a global integral

The residue theorem replaces a global accumulation around C by a finite set of local contributions:

$$\text{contour integral} = 2\pi i \sum \text{enclosed local residues.}$$

- Pole location decides whether a contribution is enclosed.
- Pole order determines the local computation method.
- The residue is the signed contribution carried by that singularity.



Common errors and quick checks

- **Wrong pole order:** simplify numerator and denominator first.
- **Wrong enclosed set:** sketch or parameterize the contour.
- **Wrong sign:** check clockwise versus counterclockwise orientation.
- **Simple-pole formula at a repeated pole:** apply derivatives instead.
- **Summing every pole:** include only enclosed poles, unless using infinity deliberately.

A direct local expansion is often the best independent check.

What we have learned

- A residue is the coefficient of $(z - z_0)^{-1}$ in a Laurent expansion.
- Limits, quotient formulas, derivatives, and short series give efficient local computations.
- The residue theorem converts a closed-contour integral into a finite sum of local contributions.
- Orientation and the enclosed set of singularities are essential.
- The residue at infinity provides a useful global balance and computational shortcut.

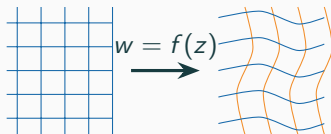
Reserved for L08

Real definite integrals, Jordan's lemma, the argument principle, and transcendental-equation applications will be developed after L07.

From residues to conformal geometry

Residues encode behavior around isolated singularities. Next we return to points where the derivative is nonzero and ask:

How does an analytic function transform angles, grids, and regions?



Next $\gg$ Lecture 07 — Conformal Mappings