L07 — Conformal Mappings

Engineering Mathematics I

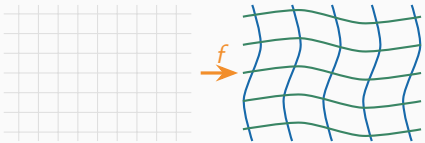
Profs. Tianyu DONG & Jiawei WANG

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School of Electrical Engineering, Xi'an Jiaotong University

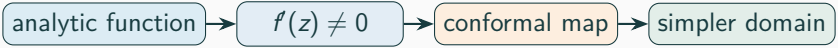
Road map

- From Analyticity to Geometry
- A Practical Library of Maps
- How to Solve a Region-Mapping Problem
- Why Conformal Maps Help Engineering



From analyticity to a geometry toolbox

In earlier lectures, analyticity was mainly a property to be tested. We now use it as a geometric design tool.



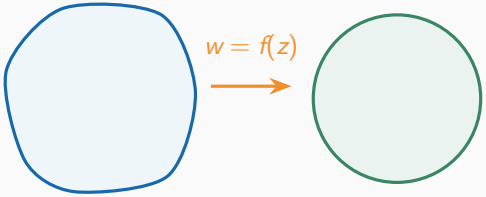
For every map we ask:

- What geometry does it create?
- What is its derivative, and where is it conformal?
- Is it one-to-one on the chosen domain?
- How do boundaries, corners, and an interior point move?
- Does the map require a branch choice?

From Analyticity to Geometry

Why map one domain to another?

Many two-dimensional engineering models are easy on a disk or half-plane but difficult on the physical domain.



Strategy

Transform the geometry $\rightarrow$ solve the simpler problem $\rightarrow$ transfer the solution back.

Mapping vocabulary: what must we track?

Let $w = f(z)$ be defined on a domain $D \subset$.

Points and sets

- $w_0 = f(z_0)$ is the *image* of z_0;
- z_0 is a *preimage* of w_0;
- $f(D) = \{f(z) : z \in D\}$ is the image domain.

Boundaries and curves

A curve $C : z = z(t)$ maps to

$$\Gamma : w = f(z(t)).$$

Boundaries, corners, and one interior point usually reveal the image region.

Important

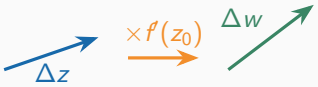
Mapping the boundary is not enough: a useful region map must also be one-to-one on the selected domain.

The derivative is a local linear map

If f is differentiable at z_0, then

$$f(z_0 + \Delta z) = f(z_0) + f'(z_0)\Delta z + o(|\Delta z|).$$

For sufficiently small Δz, the leading action is multiplication by $f'(z_0)$.

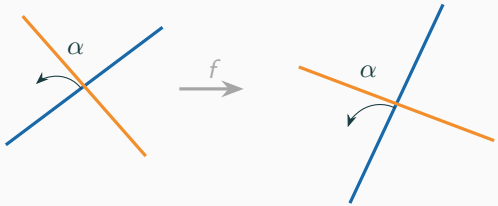


Geometric meaning

A nonzero complex multiplier applies the same local rotation and scale factor to every tangent direction.

What does “conformal” mean?

A map $w = f(z)$ is *conformal at* z_0 if it preserves the magnitude and orientation of angles between smooth curves through z_0.



A reflection preserves unsigned angles but reverses orientation; it is therefore not conformal in the analytic sense used here.

The fundamental conformality theorem

Theorem

If f is analytic near z_0 and $f'(z_0) \neq 0$, then f is conformal at z_0.

If two tangent vectors are $z_k'(0)$, then

$$w_k'(0) = f'(z_0) z_k'(0), \qquad k = 1, 2,$$

and hence

$$\arg \frac{w_2'(0)}{w_1'(0)} = \arg \frac{z_2'(0)}{z_1'(0)}.$$

Critical point

If $f'(z_0) = 0$, the linear part disappears. Higher-order terms control the local geometry, so conformality may fail.

Worked example: where is z^2 conformal?

For $f(z) = z^2$,

$$f'(z) = 2z.$$

Therefore f is conformal at every $z_0 \neq 0$.

At $z_0 = 1 + \mathrm{i}$

$$f'(z_0) = 2\sqrt{2}\,\mathrm{e}^{\mathrm{i}\pi/4}.$$

Locally: scale by $2\sqrt{2}$ and rotate by $\pi/4$.

At $z_0 = 0$

$$f(\Delta z) = (\Delta z)^2.$$

Angles are multiplied rather than shifted by one common rotation.

What exactly is preserved locally?

Write

$$f'(z_0) = \rho e^{i\phi}, \qquad \rho > 0.$$

Then $\Delta w \approx \rho e^{i\phi} \Delta z$.

Length

$$|\mathrm{d}w| \approx \rho |\mathrm{d}z|$$

Direction

$$\arg \mathrm{d}w \approx \arg \mathrm{d}z + \phi$$

Area

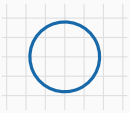
$$\mathrm{d}A_w \approx \rho^2 \mathrm{d}A_z$$

Do not over-interpret conformality

Conformal means locally shape-preserving up to scale; it does *not* mean length- or area-preserving.

Three behaviors to distinguish

Conformal



$f'(z_0) \neq 0$

Critical



$f'(z_0) = 0$

Nonanalytic distortion



$\mathrm{Re}(z)^2 + \mathrm{i}\,\mathrm{Im}(z)$

Local conformality is not global invertibility

The map $w = z^2$ has $f'(z) \neq 0$ on any annulus excluding 0, but

$$z^2 = (-z)^2.$$

Thus it is not one-to-one on a full annulus.

Local question

Is $f'(z) \neq 0$ at the point of interest?

Global question

Is f one-to-one on the entire selected domain?

Student checkpoint

A formula can be locally conformal but still be unsuitable as a region map.

Summary: local conformality

Question	Answer
What is the local model?	$f(z_0 + \Delta z) \approx f(z_0) + f'(z_0)\Delta z$.
What does $f'(z_0)$ do?	It rotates by $\arg f'(z_0)$ and scales lengths by $\lvert f'(z_0)\rvert$.
When is the map conformal?	If f is analytic near z_0 and $f'(z_0) \neq 0$.
What can fail?	Critical points $f'(z_0) = 0$, branch points, singularities, or loss of global injectivity.
What must a region map satisfy?	Local conformality plus one-to-one behavior on the selected domain.

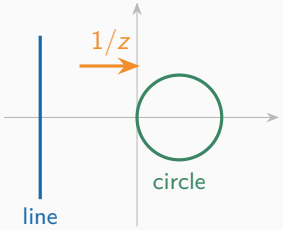
Main message

Local angle preservation and global invertibility are different questions; a valid region map needs both.

A Practical Library of Maps

Key points

- Translation, rotation, dilation, and inversion
- Möbius transformations and generalized circles
- Three-point construction
- Power and square-root maps for sectors and slit planes
- Exponential maps for strips
- Choosing a map from boundary geometry



Start with four elementary operations

Translation	Rotation	Dilation	Inversion
$w = z + \beta$	$w = \mathrm{e}^{\mathrm{i}\theta} z$	$w = \rho z$	$w = 1/z$

The first three are similarities. Inversion is the new geometric ingredient: it exchanges near and far points and converts many lines into circles.

Inversion: first understand radius and angle

For $z = re^{i\theta} \neq 0$,

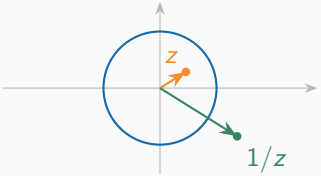
$$w = \frac{1}{z} = \frac{1}{r}e^{-i\theta}.$$

Radial action

$r \mapsto r^{-1}$: inside and outside the unit circle are exchanged.

Angular action

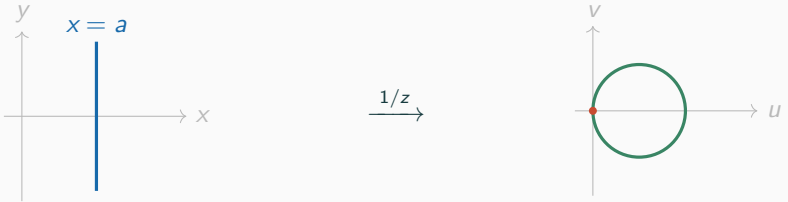
$\theta \mapsto -\theta$, while $f'(z) = -1/z^2 \neq 0$ for $z \neq 0$.



Inversion example: a line becomes a circle

Let $w = u + iv = 1/z$. Since $z = 1/w$,

$$\operatorname{Re} z = a > 0 \iff \frac{u}{u^2 + v^2} = a \iff \boxed{\left(u - \frac{1}{2a}\right)^2 + v^2 = \frac{1}{4a^2}}.$$



The image circle passes through $w = 0$, corresponding to the point at infinity on the original line.

Möbius transformations: the circle–line machine

A linear fractional transformation is

$$T(z) = \frac{az + b}{cz + d}, \qquad ad - bc \neq 0.$$

Its derivative is

$$T'(z) = \frac{ad - bc}{(cz + d)^2},$$

so it is conformal wherever finite.

Geometry

Circles and lines in $\widehat{}$ map to circles or lines.

Global behavior

They are one-to-one and onto on the extended complex plane $\widehat{}$.

Why students should care

Once the boundary is made of lines and circles, a Möbius map is usually the first tool to try.

Three points determine a Möbius map

Three distinct source points and three distinct target points determine a unique Möbius transformation.

Example

Require

$$0 \mapsto -1, \qquad 1 \mapsto 0, \qquad \infty \mapsto 1.$$

The resulting map is

$$\boxed{T(z) = \frac{z-1}{z+1}}.$$

Efficient construction

Choose special points such as 0, 1, and ∞ whenever possible; then test one interior point to determine the correct side of the mapped boundary.

Canonical Möbius maps to remember

Right half-plane $\to$ disk

$$w = \frac{z-1}{z+1}.$$

The boundary $\operatorname{Re} z = 0$ maps to $|w| = 1$, and $z = 1$ maps to $w = 0$.

Upper half-plane $\to$ disk

$$w = \frac{z-\mathrm{i}}{z+\mathrm{i}}.$$

The boundary $\operatorname{Im} z = 0$ maps to $|w| = 1$, and $z = \mathrm{i}$ maps to $w = 0$.



Power maps open and close angles

On a fixed branch,

$$w = z^{\alpha} = r^{\alpha} e^{i\alpha\theta}.$$

Therefore the sector

$$0 < \arg z < \beta$$

maps onto the upper half-plane under

$$\boxed{w = z^{\pi/\beta}}.$$



Branch choice is part of the geometry, not just an algebraic convention.

Square roots straighten slit planes

Choose the principal branch $-\pi < \arg z < \pi$. Then

$$w = \sqrt{z} = \sqrt{r}\,\mathrm{e}^{\mathrm{i}\theta/2}, \qquad -\frac{\pi}{2} < \arg w < \frac{\pi}{2}.$$

Hence

$$\setminus\{x \in \mathbb{R} : x \le 0\} \quad \xrightarrow{\sqrt{z}} \quad \{\mathrm{Re}\, w > 0\}.$$

Interpretation

The branch cut makes the square root one-to-one; the two sides of the cut become the two sides of the imaginary-axis boundary.

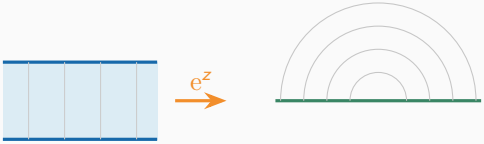
Exponential maps turn strips into sectors or half-planes

For $z = x + iy$,

$$w = e^z = e^x e^{iy}.$$

Thus

- horizontal lines $y =$ constant map to rays;
- vertical lines $x =$ constant map to circles;
- the strip $0 < \operatorname{Im} z < \pi$ maps onto the upper half-plane.



A compact map-selection table

Source geometry	First map to try	Typical target
sector of angle β	$z^{\pi/\beta}$	upper half-plane
horizontal strip	e^{z}	sector / half-plane
upper half-plane	$(z-\mathrm{i})/(z+\mathrm{i})$	unit disk
right half-plane	$(z-1)/(z+1)$	unit disk
circle/line boundary	Möbius map	circle/line boundary
slit plane	root / logarithm	half-plane / strip

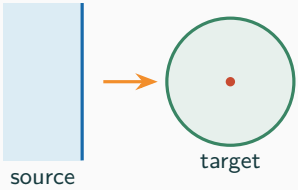
Design principle

Choose the map from the *boundary geometry*, not from the algebraic appearance of the formula.

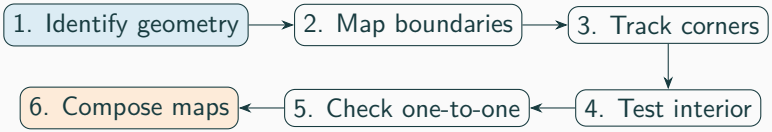
How to Solve a Region-Mapping Problem

Key points

- Read the source geometry before choosing a formula
- Map boundary components and special points first
- Compose standard maps rather than inventing one complicated map
- Test one interior point to determine the correct side
- Check one-to-one behavior on the selected domain
- Check topology before doing unnecessary algebra



A reliable workflow: use this on every problem



Common mistake

A mapped boundary tells you a curve, not automatically which side is the image. Always test an interior point.

Example I: quadrant $\to$ half-plane

Let

$$Q = \{z : 0 < \arg z < \pi/2\}.$$

Try the power map $\zeta = z^2$ because it doubles angles:

$$0 < \arg \zeta = 2 \arg z < \pi.$$

Therefore Q maps one-to-one onto the upper half-plane.



Example II: quadrant $\to$ disk by composition

We already know

$$Q \xrightarrow{\zeta = z^2} \{\operatorname{Im}\zeta > 0\}.$$

Follow it with the Cayley map:

$$\{\operatorname{Im}\zeta > 0\} \xrightarrow{w=(\zeta-\mathrm{i})/(\zeta+\mathrm{i})} \{|w| < 1\}.$$

Hence

$$\boxed{w = \frac{z^2 - \mathrm{i}}{z^2 + \mathrm{i}}}.$$

Verification

The two boundary rays map to $|w| = 1$, while $z = \mathrm{e}^{\mathrm{i}\pi/4}$ maps to $w = 0$. The map is one-to-one on the open first quadrant.

Example III: strip $\to$ disk — construct first

Let

$$S = \{z : 0 < \operatorname{Im} z < \pi\}.$$

The geometry suggests two standard steps:

$$S \xrightarrow{\zeta = e^z} \{\operatorname{Im} \zeta > 0\} \xrightarrow{w=(\zeta-i)/(\zeta+i)} \{|w| < 1\}.$$

Therefore

$$\boxed{w = \frac{e^z - i}{e^z + i}}.$$

What to learn from this example

Do not search for one complicated formula. Reduce the geometry through a sequence of familiar canonical domains.

Example III: now verify the complete region map

Track the two boundary components under $\zeta = e^z$:

- $z = x$ maps to $\zeta = e^x > 0$;
- $z = x + i\pi$ maps to $\zeta = -e^x < 0$;
- the Cayley map sends the complete real axis to $|w| = 1$.

Test one interior point:

$$z = i\pi/2 \mapsto \zeta = i \mapsto w = 0.$$

Finally, e^z is one-to-one on this strip and the Cayley map is one-to-one.

Verification pattern

Boundary correspondence + one interior point + one-to-one behavior determines the image region.

Composition is the main design tool

Build a difficult map as

$$D_0 \xrightarrow{f_1} D_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} D_n.$$

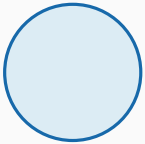
At every stage ask four questions:

1. What happens to each boundary component?
2. Are branch choices consistent?
3. Is the map one-to-one on this domain?
4. Which side of the mapped boundary is the image?

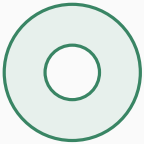
If every f_k is conformal on the relevant domain, then the composition is conformal there.

Before doing algebra: check the topology

A one-to-one conformal equivalence preserves simple connectivity. A hole cannot disappear under a one-to-one analytic change of coordinates.



simply connected



doubly connected

Disk-type target

No hole: disk or half-plane is a plausible canonical target.

Annulus-type target

One hole remains one hole; it cannot map one-to-one onto a disk.

What the Riemann mapping theorem tells us

Riemann mapping theorem

Every nonempty simply connected proper domain $D \subsetneq$ is conformally equivalent to the unit disk.

It guarantees

For a very large class of planar domains, a canonical disk model exists.

It does not give

It does not automatically provide an explicit formula, nor does it cover multiply connected domains in this form.

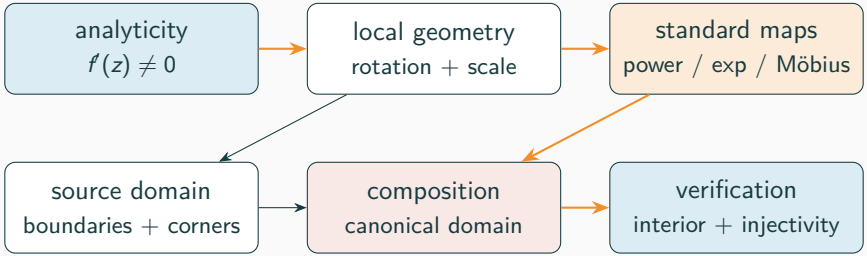
In engineering calculations, explicit elementary maps and compositions remain the practical first choice.

Practice

1. Determine where $f(z) = z^3$ is conformal, and give its local scale and rotation at $z = 1 + \mathrm{i}$.
2. Map the sector $0 < \arg z < \pi/3$ onto the upper half-plane.
3. Find a Möbius map that sends the right half-plane onto the unit disk and verify the boundary and one interior point.
4. Construct a map from the strip $0 < \operatorname{Im} z < \pi/2$ onto the unit disk.
5. Explain why z^2 cannot be one-to-one on a full annulus centered at the origin.

What we have learned & Concept map

- Analyticity with $f'(z) \neq 0$ gives local rotation and dilation, hence conformality.
- Standard maps solve standard geometries: sectors, strips, half-planes, disks, lines, and circles.
- Difficult region maps are usually compositions of simple maps.
- Boundary correspondence + one interior point + injectivity determines the image region.



Why Conformal Maps Help Engineering

Key points

- Harmonic conjugates generate orthogonal coordinates
- Conformal maps preserve harmonicity
- Complex potentials encode equipotentials and streamlines
- Canonical-domain solutions can be transferred back
- Electrostatics, steady heat flow, and ideal flow share the same Laplace structure



Analytic functions generate orthogonal coordinates

If $f = u + \mathrm{i}v$ is analytic, the Cauchy–Riemann equations give

$$\nabla u = (u_x, u_y), \qquad \nabla v = (-u_y, u_x).$$

Hence

$$\nabla u \cdot \nabla v = 0, \qquad |\nabla u| = |\nabla v| = |f'(z)|.$$

Where $f'(z) \neq 0$, the level curves $u = c_1$ and $v = c_2$ intersect orthogonally.

Geometric interpretation

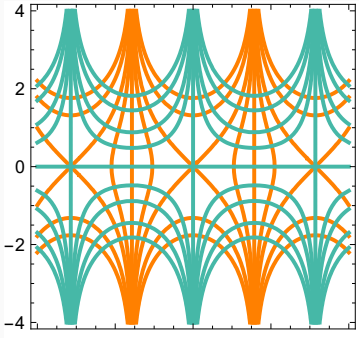
The real and imaginary parts of one analytic function form a natural curvilinear coordinate grid.

Read a conformal grid: two examples

$f(z) = \cos z$

$$f(z) = \cos x \cosh y - \mathrm{i} \sin x \sinh y.$$

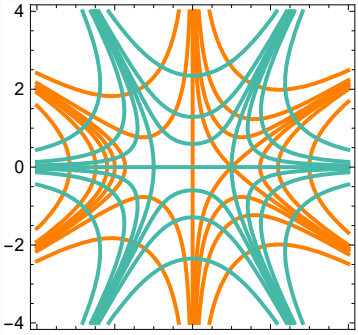
Level curves of $\mathrm{Re}\, f$ and $\mathrm{Im}\, f$ meet orthogonally away from critical points.



$f(z) = z^3 - 3z + 2$

$$f'(z) = 3(z^2 - 1).$$

The grid degenerates precisely at $z = \pm 1$, where $f'(z) = 0$.



Conformal maps preserve harmonicity

Let $U(\xi, \eta)$ be harmonic in a w-domain and let $w = f(z)$ be analytic. Then

$$\Delta_z(U \circ f) = |f'(z)|^2 (\Delta_w U) \circ f.$$

Therefore

$$\Delta_w U = 0 \quad \Longrightarrow \quad \Delta_z(U \circ f) = 0.$$

Engineering significance

Electrostatic potential, steady temperature, and ideal incompressible potential flow can be transferred between conformally related domains.

Complex potential: geometry becomes physics

Write the complex potential as

$$F(z) = \phi(x, y) + \mathrm{i}\psi(x, y).$$

Velocity potential

ϕ is harmonic, $\mathbf{v} = \nabla\phi$, and $\phi =$ constant gives equipotential curves.

Stream function

ψ is the harmonic conjugate of ϕ, and $\psi =$ constant gives streamlines.

The two families intersect orthogonally wherever $F'(z) \neq 0$.

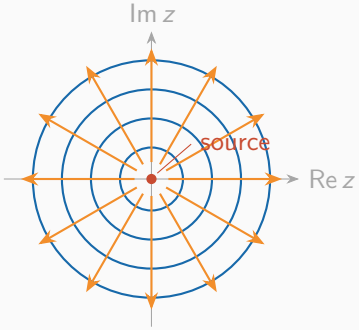
Worked example: a point source

For

$$F(z) = \operatorname{Log} z = \log r + \mathrm{i}\theta,$$

we have $\phi = \log r$ and $\psi = \theta$.

- equipotentials: $r =$ constant;
- streamlines: $\theta =$ constant;
- speed: $|F'(z)| = 1/r$;
- singularity at $z = 0$.

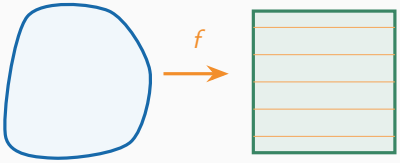


Transfer a canonical solution back to the physical domain

Suppose $\zeta = f(z)$ maps the physical domain conformally onto a simple canonical domain, and $W(\zeta)$ is known there. Then

$$\boxed{F(z) = W(f(z))}$$

gives the transferred complex potential.



This closes the loop

The lecture began by simplifying geometry; harmonicity tells us why the corresponding physical solution can be transported back.

Engineering glimpse: the Joukowski map

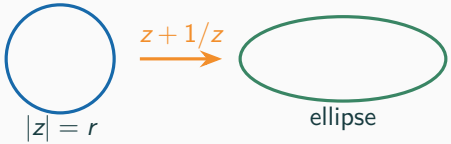
For

$$w = z + \frac{1}{z}, \qquad z = re^{i\theta},$$

we obtain

$$w = (r + r^{-1})\cos\theta + i(r - r^{-1})\sin\theta.$$

Thus centered circles $|z| = r > 1$ become ellipses, while $r = 1$ collapses to the segment $[-2, 2]$.



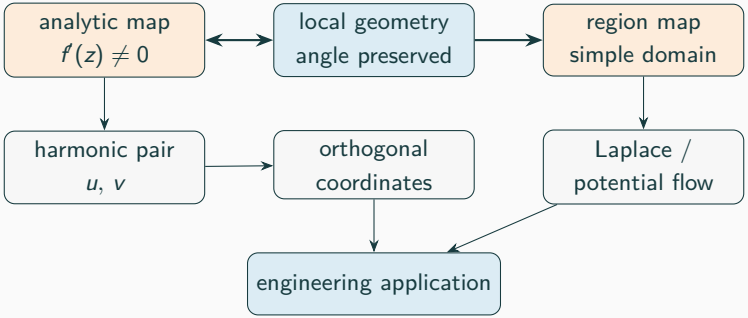
Off-centered circles lead to the classical airfoil construction studied in later applications.

Practice

1. For $F(z) = \operatorname{Log} z$, identify the equipotential curves and streamlines, and explain why they are orthogonal.
2. If $U(\xi, \eta)$ is harmonic and $w = f(z)$ is conformal, explain why $U \circ f$ remains harmonic.
3. Sketch how a solution on the unit disk can be transferred to a physical domain using a conformal map.
4. For the Joukowski map $w = z + 1/z$, identify the critical points and explain the degeneration of the unit circle.

What we have learned & Concept map

- Conformal maps simplify geometry without changing local angles.
- The same analytic structure produces orthogonal coordinates and preserves harmonic fields.
- Region mapping is therefore not only geometric: it transfers Laplace and potential-flow problems between domains.



Summary

- Analytic maps with $f'(z_0) \neq 0$ are locally a rotation and dilation, hence conformal.
- Local conformality does not guarantee global one-to-one behavior.
- Möbius, power, root, exponential, and logarithmic maps form a practical geometry-to-geometry library.
- Region mapping should be verified by boundary correspondence, an interior test point, and injectivity.
- Composition reduces difficult domains to canonical domains step by step.
- Conformal maps preserve harmonic structure, which is why they are useful in electrostatics, heat flow, and potential flow.

Looking ahead to applications of complex analysis

L08 will use the mapping tools developed here to solve physical problems on simpler domains.



The key skill is not memorising formulas, but recognising geometry and composing the right elementary maps.

Next $\gg$ Lecture 08 — Applications of Complex Analysis