L08 — Applications of Complex Analysis

Engineering Mathematics I

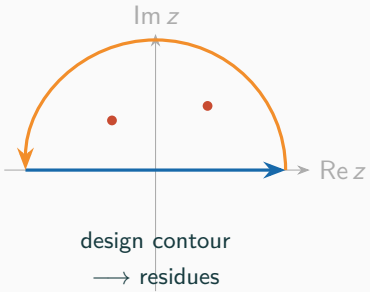
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Fall 2026

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Road map

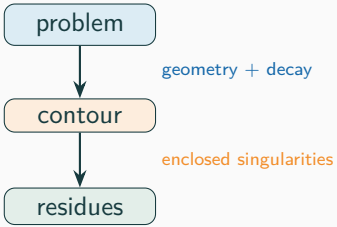
- From Residues to Contour Design
- Evaluation of Real Integrals
- Argument Principle and Root Counting
- Special Functions and Engineering Interpretation



From Residues to Contour Design

Key points

- Turn a real problem into a complex contour problem
- Let geometry, decay, poles, and branch cuts determine the contour
- Separate the desired integral from auxiliary contour pieces
- Use residues only after the contour and its orientation are fixed

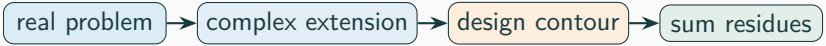


From a theorem to a problem-solving toolbox

L06 established the residue theorem,

$$\oint_C f(z)\,\mathrm{d}z = 2\pi\mathrm{i} \sum_{z_k \in \operatorname{int} C} \operatorname{Res}(f, z_k).$$

In this lecture the main question changes from *"Can we compute a contour integral?"* to **"How should we design the contour so that it solves the problem we actually want?"**



Main message

Do not start by computing residues. First ask: *which contour turns the given problem into a controllable contour integral?*

Residues turn contour design into an algorithm

- **What do we want?** Identify the real integral, root count, or phase change to recover.
- **What is the complex extension?** Write a suitable analytic or meromorphic function.
- **What obstructs analyticity?** Locate poles, branch points, and behavior at infinity.
- **Which contour helps?** Choose the contour and orientation from geometry and decay.
- **What vanishes or jumps?** Estimate large/small arcs and branch edges.
- **Which residues contribute?** Sum only enclosed singularities and solve for the target quantity.

Most common failure

Correct residues on the wrong contour still give the wrong answer.

Contour choice is a consequence, not a rule to memorize

Example A: algebraic decay

$$I_1 = \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^2+1}.$$

On a large semicircle, the integrand is $O(R^{-2})$ while the arc length is $O(R)$.

Question: can the large arc vanish?

Example B: exponential factor

$$I_2 = \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}\omega x}}{x^2+1}\,\mathrm{d}x.$$

Now the decisive factor is $|\mathrm{e}^{\mathrm{i}\omega(x+\mathrm{i}y)}| = \mathrm{e}^{-\omega y}$.

Question: which half-plane gives decay?

Learning strategy

Do not memorize contour names first. Work through representative examples, identify the feature that controls the contour, and summarize the pattern afterward.

Optional shortcut: residue at infinity

For a meromorphic function with finitely many singularities,

$$\operatorname{Res}(f,\infty) = -\operatorname{Res}\left(\frac{1}{w^2} f\left(\frac{1}{w}\right); 0\right),$$

and

$$\operatorname{Res}(f,\infty) + \sum_k \operatorname{Res}(f, z_k) = 0.$$

Therefore a contour enclosing every finite pole satisfies

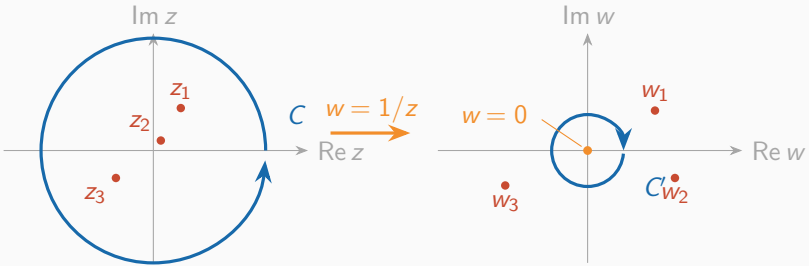
$$\oint_C f(z)\,\mathrm{d}z = -2\pi\mathrm{i}\operatorname{Res}(f,\infty).$$

When should we use it?

Use it only when the expansion at infinity is simpler than summing the finite residues. It is a shortcut, not a new contour rule.

Inversion makes infinity local

The substitution $w = 1/z$ maps a large counterclockwise contour in the z-plane to a small clockwise contour around $w = 0$.



Therefore $\operatorname{Res}(f, \infty)$ is computed from the ordinary residue at $w = 0$, while the reversed orientation supplies the minus sign.

Worked example: choosing infinity strategically

Let

$$f(z) = \frac{8z - 1}{(z-1)(z+4)}.$$

A contour enclosing both finite poles may be evaluated from the coefficient of $1/z$ at infinity:

$$f(z) = \frac{8}{z} + O\left(\frac{1}{z^2}\right), \qquad \operatorname{Res}(f, \infty) = -8.$$

Hence

$$\oint_C f(z)\,\mathrm{d}z = -2\pi\mathrm{i}(-8) = \boxed{16\pi\mathrm{i}}.$$

This reproduces the sum of the residues at 1 and -4 with less algebra.

Summary: contour design

Question	What to check
What quantity is required?	Identify the real integral, phase change, or root count that must appear on the contour.
Where are the singularities?	Locate poles, poles on the path, and branch points before choosing a contour.
Where does the integrand decay?	Use algebraic decay, exponential decay, or an ML/Jordan estimate to control auxiliary arcs.
Is a branch required?	Choose the branch first; the branch cut then constrains the contour.
Which residues enter?	Sum only singularities enclosed by the chosen orientation.

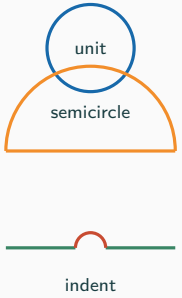
Main message

Choose the contour from analytic structure first; compute residues second.

Evaluation of Real Integrals

Key points

- Full-period trigonometric integrals $\rightarrow$ unit circle
- Rational real-line integrals $\rightarrow$ large semicircle
- Oscillatory integrals $\rightarrow$ the half-plane of exponential decay
- Poles on the path $\rightarrow$ indentation and principal value
- Branch powers $\rightarrow$ keyhole contour



Periodic trigonometric integrals

For integrals over a full period, set

$$z = \mathrm{e}^{\mathrm{i}\theta}, \quad \mathrm{d}\theta = \frac{\mathrm{d}z}{\mathrm{i}z}, \quad \cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right), \quad \sin\theta = \frac{1}{2\mathrm{i}}\left(z - \frac{1}{z}\right).$$

Then

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\,\mathrm{d}\theta = \oint_{|z|=1} \widetilde{R}(z)\,\mathrm{d}z.$$

where R and $\widetilde{R}$ represent rational functions before and after integral transformation.

What must still be checked?

Simplify the rational function, locate every pole, and retain only poles strictly inside the unit circle.

Worked example: a full-period integral

Evaluate

$$I = \int_0^{2\pi} \frac{\mathrm{d}\theta}{5 + 4\cos\theta}.$$

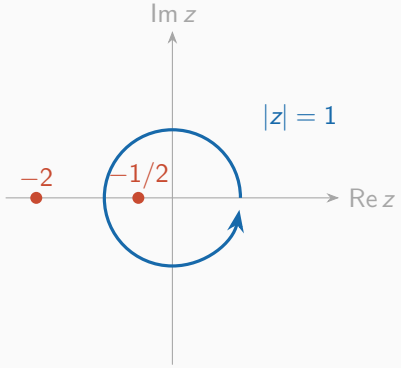
The unit-circle substitution gives

$$I = -\mathrm{i} \oint_{|z|=1} \frac{\mathrm{d}z}{(z+2)(2z+1)}.$$

Only $z = -1/2$ is enclosed. Since

$$\operatorname{Res}\left(\frac{1}{(z+2)(2z+1)}; -\frac{1}{2}\right) = \frac{1}{3},$$

$$I = -\mathrm{i}(2\pi\mathrm{i})\frac{1}{3} = \boxed{\frac{2\pi}{3}}.$$



- $z = -1/2$ lies inside $|z| = 1$;
- $z = -2$ lies outside;
- positive orientation gives $+2\pi\mathrm{i}$.

Rational integrals on the real axis

For a rational function $R(x) = P(x)/Q(x)$ with no real poles, a semicircular contour is effective when

$$\deg Q \geq \deg P + 2.$$

If C_R is the upper semicircle, then

$$\int_{-R}^{R} R(x)\,\mathrm{d}x + \int_{C_R} R(z)\,\mathrm{d}z = 2\pi\mathrm{i} \sum_{\operatorname{Im} z_k > 0} \operatorname{Res}(R; z_k).$$

The degree condition is a convenient sufficient condition for the arc integral to vanish; sharper estimates may permit slower decay.

Why the large arc disappears

Suppose $|R(z)| \le M_R$ on C_R. The ML estimate gives

$$\left| \int_{C_R} R(z)\,\mathrm{d}z \right| \le \pi R M_R.$$

Hence the auxiliary arc vanishes whenever

$$RM_R \longrightarrow 0 \qquad (R \to \infty).$$

Common mistake

"The integrand tends to zero" is not sufficient by itself: the arc length grows like R.

Worked example: a rational integral

Evaluate

$$I = \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^4 + 1}.$$

The upper-half-plane poles are

$$z_1 = \mathrm{e}^{\mathrm{i}\pi/4}, \qquad z_2 = \mathrm{e}^{3\mathrm{i}\pi/4},$$

and, because the poles are simple,

$$\operatorname{Res}\left(\frac{1}{z^4+1}; z_k\right) = \frac{1}{4z_k^3}.$$

On C_R, the integrand is $O(R^{-4})$ while the arc length is $O(R)$, so the arc contribution is $O(R^{-3}) \to 0$.

Worked example: summing the two residues

Using $z_1 = \mathrm{e}^{\mathrm{i}\pi/4}$ and $z_2 = \mathrm{e}^{3\mathrm{i}\pi/4}$,

$$\operatorname{Res}(f, z_1) + \operatorname{Res}(f, z_2) = \frac{1}{4}\left(\mathrm{e}^{-3\pi\mathrm{i}/4} + \mathrm{e}^{-\pi\mathrm{i}/4}\right) = -\frac{\mathrm{i}}{2\sqrt{2}}.$$

Therefore

$$I = 2\pi\mathrm{i}\left(-\frac{\mathrm{i}}{2\sqrt{2}}\right) = \boxed{\frac{\pi}{\sqrt{2}}}.$$

Since the integrand is even, this also gives $\int_0^\infty (1+x^4)^{-1}\,\mathrm{d}x = \pi/(2\sqrt{2})$.

Fourier-type integrals and Jordan's lemma

To evaluate

$$\int_{-\infty}^{\infty} R(x)\mathrm{e}^{\mathrm{i}\omega x}\,\mathrm{d}x,$$

close the contour where

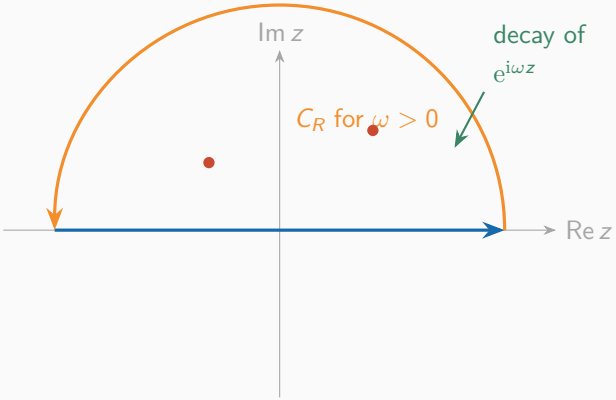
$$|\mathrm{e}^{\mathrm{i}\omega(x+\mathrm{i}y)}| = \mathrm{e}^{-\omega y}$$

decays.

$\omega > 0$	$\omega < 0$
Close in the upper half-plane.	Close in the lower half-plane.

Jordan's lemma supplies conditions under which the large semicircle contributes zero.

Contour choice follows exponential decay



Enclose the poles in the decay half-plane;
do not choose the semicircle before checking the sign of ω.

Worked example: a Fourier integral

For $a > 0$, first take $\omega > 0$ and consider

$$J(\omega) = \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}\omega x}}{x^2 + a^2}\,\mathrm{d}x.$$

Closing upward encloses only $z = \mathrm{i}a$, where

$$\operatorname{Res}\left(\frac{\mathrm{e}^{\mathrm{i}\omega z}}{z^2 + a^2}; \mathrm{i}a\right) = \frac{\mathrm{e}^{-a\omega}}{2\mathrm{i}a}.$$

For $\omega < 0$, close in the lower half-plane. Combining both signs of ω and taking the real part gives

$$\boxed{\int_{-\infty}^{\infty} \frac{\cos(\omega x)}{x^2 + a^2}\,\mathrm{d}x = \frac{\pi}{a}\mathrm{e}^{-a|\omega|}}.$$

From the complex integral to sine and cosine parts

Writing $\mathrm{e}^{\mathrm{i}\omega x} = \cos(\omega x) + \mathrm{i}\sin(\omega x)$ gives

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}\omega x}}{x^2 + a^2}\,\mathrm{d}x = \int_{-\infty}^{\infty} \frac{\cos(\omega x)}{x^2 + a^2}\,\mathrm{d}x,$$

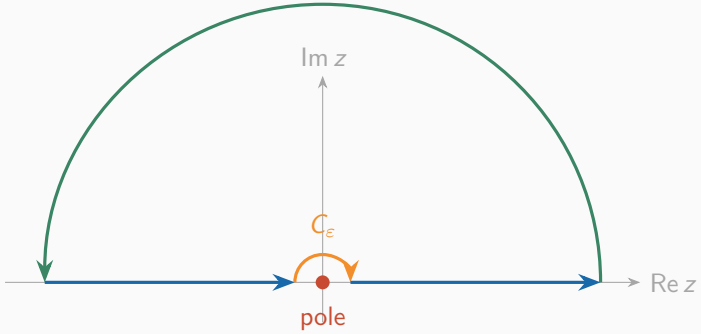
because the sine part is odd. Thus the residue calculation also reveals the Fourier transform pair

$$\frac{1}{x^2 + a^2} \quad \longleftrightarrow \quad \frac{\pi}{a}\mathrm{e}^{-a|\omega|}.$$

This is the bridge from contour integration to frequency-domain analysis in L09.

Poles on the real axis require indentation

A pole on the path requires a principal-value interpretation and a small detour. Here PV denotes the *Cauchy principal value*: the singular point is approached symmetrically from the two sides.



The small arc contributes a fraction of $2\pi i \operatorname{Res}(f; z_0)$ determined by its angle and orientation.

A semicircular indentation contributes half a residue

Near a simple pole x_0 write

$$f(z) = \frac{a_{-1}}{z - x_0} + h(z).$$

On an upper indentation traversed clockwise, $z = x_0 + \varepsilon \mathrm{e}^{\mathrm{i}\theta}$ with $\theta : \pi \to 0$, so

$$\int_{C_\varepsilon} f(z)\,\mathrm{d}z \longrightarrow -\mathrm{i}\pi a_{-1} = -\mathrm{i}\pi \operatorname{Res}(f, x_0).$$

The sign depends on both the side of indentation and its orientation.

Worked example: the Dirichlet integral

Consider $f(z) = \mathrm{e}^{\mathrm{i}z}/z$ and indent above the pole at $z = 0$. Since $\operatorname{Res}(f, 0) = 1$, the upper clockwise indentation contributes

$$\int_{C_\varepsilon} f(z)\,\mathrm{d}z \longrightarrow -\mathrm{i}\pi.$$

No poles remain inside the indented upper contour, so

$$\operatorname{PV}\int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}x}}{x}\,\mathrm{d}x = \mathrm{i}\pi.$$

Now separate real and imaginary parts:

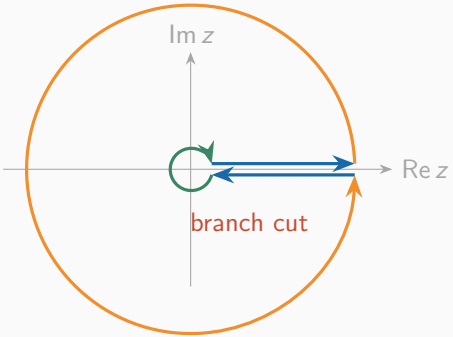
$$\operatorname{PV}\int_{-\infty}^{\infty} \frac{\cos x}{x}\,\mathrm{d}x + \mathrm{i}\int_{-\infty}^{\infty} \frac{\sin x}{x}\,\mathrm{d}x = \mathrm{i}\pi.$$

The cosine term is odd, while $\sin x/x$ is even. Therefore

$$2\int_0^{\infty} \frac{\sin x}{x}\,\mathrm{d}x = \pi, \qquad \boxed{\int_0^{\infty} \frac{\sin x}{x}\,\mathrm{d}x = \frac{\pi}{2}}.$$

Keyhole contours convert a branch jump into a factor

For $0 < \alpha < 1$, use $z^{\alpha-1} = \exp((\alpha-1)\operatorname{Log} z)$ with $0 < \arg z < 2\pi$.



The values above and below the cut differ by $\mathrm{e}^{2\pi\mathrm{i}(\alpha-1)}$.

Worked example: a beta-type integral

For $0 < \alpha < 1$,

$$I(\alpha) = \int_0^\infty \frac{x^{\alpha-1}}{1+x}\,\mathrm{d}x.$$

The keyhole contour encloses $z = -1$, and the branch jump gives

$$\left(1 - \mathrm{e}^{2\pi \mathrm{i}\alpha}\right) I(\alpha) = 2\pi \mathrm{i}\,\operatorname{Res}\left(\frac{z^{\alpha-1}}{1+z}; -1\right).$$

With $\arg(-1) = \pi$,

$$\boxed{I(\alpha) = \frac{\pi}{\sin(\pi\alpha)}}.$$

This identity will reappear through the gamma function.

Why the keyhole restrictions matter

For $0 < \operatorname{Re}\alpha < 1$,

$$\frac{z^{\alpha-1}}{1+z} = O\big(R^{\operatorname{Re}\alpha-2}\big) \quad (|z| = R),$$

so the outer arc is $O(R^{\operatorname{Re}\alpha-1}) \to 0$. Near the origin,

$$\frac{z^{\alpha-1}}{1+z} = O\big(\varepsilon^{\operatorname{Re}\alpha-1}\big)\,,$$

and the inner arc is $O(\varepsilon^{\operatorname{Re}\alpha}) \to 0$.

Meaning of the strip

The same inequalities that remove both arcs produce the natural convergence condition $0 < \operatorname{Re}\alpha < 1$.

Symmetry can halve the residue work

If $R(x)$ is even and decays sufficiently fast,

$$\int_{-\infty}^{\infty} R(x)\,\mathrm{d}x = 2\int_{0}^{\infty} R(x)\,\mathrm{d}x.$$

Before computing residues:

- check parity and conjugate symmetry;
- factor the denominator geometrically;
- simplify removable factors;
- use a residue at infinity only when it shortens the sum.

Algebraic simplification comes before pole formulas.

Summary: the real-integral toolkit

Integral structure	Typical contour	Decisive check
full-period trigonometric	unit circle	poles inside $\lvert z\rvert = 1$
rational on $(-\infty, \infty)$	semicircle	large-arc decay
Fourier type	decay semicircle	sign of ω / Jordan
pole on integration path	indentation	principal value + orientation
branch power $x^{\alpha-1}$	keyhole	branch jump + two arc limits

Main message

The integrand tells you which contour to use; there is no single "residue contour" for every real integral.

Practice

Pause before calculating

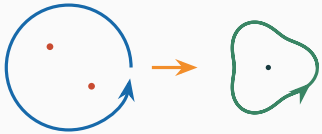
For each problem, first state the contour and the single condition that makes it work. Only then start residue calculations.

1. Which contour would you choose for $\displaystyle\int_0^{2\pi} \frac{\mathrm{d}\theta}{3+2\cos\theta}$? State the substitution and the pole test.
2. For $\displaystyle\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^6+1}$, identify the poles enclosed by an upper semicircle and explain why the arc vanishes.
3. For $\displaystyle\int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}\omega x}}{x^2+4}\,\mathrm{d}x$, decide the closing half-plane for $\omega > 0$ and $\omega < 0$.
4. Explain why a pole on the real axis or a branch point at 0 requires a contour different from the ordinary semicircle.

Argument Principle and Root Counting

Key points

- Convert zeros and poles into residues of f'/f
- Interpret root counting as the winding of $f(C)$ around 0
- Use the argument principle when phase information is accessible
- Use Rouché's theorem when one term dominates on the boundary
- Distinguish root *counting* from root *location*



The logarithmic derivative detects zeros and poles

If f has a zero of order m at a, then

$$f(z) = (z-a)^m g(z), \qquad g(a) \neq 0,$$

and

$$\frac{f'(z)}{f(z)} = \frac{m}{z-a} + \frac{g'(z)}{g(z)}.$$

Thus f'/f has residue $+m$ at a zero and residue $-n$ at a pole of order n.

Local-to-global bridge

Integrating f'/f counts zeros minus poles inside a contour.

Argument principle

Let f be meromorphic inside and on a positively oriented simple closed contour C, with no zeros or poles on C. Then

$$\boxed{\frac{1}{2\pi \mathrm{i}} \oint_C \frac{f'(z)}{f(z)}\,\mathrm{d}z = N - P},$$

where N and P count zeros and poles with multiplicity. Equivalently,

$$\Delta_C \arg f(z) = 2\pi(N - P).$$

The image curve $f(C)$ winds around the origin exactly $N - P$ times.

Why the integral measures change of phase

Along C, locally choose a continuous logarithm of f:

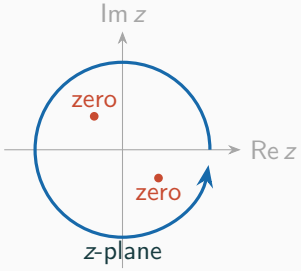
$$\frac{f'(z)}{f(z)}\,\mathrm{d}z = \mathrm{d}\log f(z) = \mathrm{d}\log|f(z)| + \mathrm{i}\,\mathrm{d}\arg f(z).$$

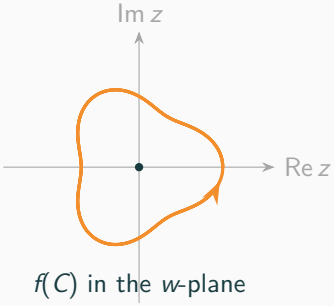
After one closed traversal, $|f|$ returns to its initial value, while the argument may change by $2\pi k$. Thus

$$\frac{1}{2\pi\mathrm{i}}\oint_C \frac{f'}{f}\,\mathrm{d}z = \frac{\Delta_C \arg f}{2\pi} = k.$$

Residue theory identifies this winding number as $N - P$.

Root counting is a winding-number problem





How to read the picture

Traverse C once counterclockwise $\to$ follow the point $f(z)$ in the w-plane $\to$ count the net turns of $f(C)$ around 0.

How to count roots numerically from phase

Sample $z(\theta)$ around C and track a continuously unwrapped phase

$$\phi(\theta) = \arg f(z(\theta)).$$

Then

$$N - P = \frac{\phi(2\pi) - \phi(0)}{2\pi}.$$

- refine sampling near rapid phase changes;
- verify $f(z) \neq 0, \infty$ on the contour;
- do not use the principal argument without phase unwrapping.

This connects the analytic theorem with robust numerical root counting.

Worked example: counting polynomial zeros

For

$$p(z) = z^6 - 3z^5 + 2z^3 - z + 4,$$

sampling $p(\mathrm{e}^{\mathrm{i}\theta})$ around $|z| = 1$ and unwrapping its phase gives a total phase change of 4π. Hence

$$\frac{1}{2\pi\mathrm{i}} \oint_{|z|=1} \frac{p'(z)}{p(z)}\,\mathrm{d}z = \frac{4\pi}{2\pi} = 2.$$

Since a polynomial has no poles and p has no zeros on the contour,

$$\boxed{p \text{ has two zeros in } |z| < 1, \text{ counted with multiplicity.}}$$

The conclusion is a count, not a numerical location of the roots.

Rouché's theorem: compare with a dominant function

If f and g are analytic inside and on C, and

$$|g(z)| < |f(z)| \qquad (z \in C),$$

then f and $f+g$ have the same number of zeros inside C, counted with multiplicity.

Interpretation

Along the boundary, the perturbation g is too small to change the winding number.

Choose f as the term whose modulus is easiest to control on the boundary.

Worked example: count roots without finding them

How many zeros does

$$p(z) = z^5 + 4z + 1$$

have in $|z| < 2$? On $|z| = 2$,

$$|z^5| = 32, \qquad |4z + 1| \le 9 < 32.$$

By Rouché's theorem, $p(z)$ and z^5 have the same number of zeros in the disk. Therefore

$$\boxed{p \text{ has five zeros in } |z| < 2.}$$

No explicit root formula is needed.

Argument principle or Rouché?

Method	Best when	Main check
argument principle	phase or f'/f is accessible	no zero or pole on C
Rouché	one term dominates on C	strict inequality everywhere
direct solution	low-degree algebraic equation	reliable explicit roots
numerical roots	locations are required	verify the search region

Counting methods answer a different question from root-finding methods.

Transcendental equations use the same logic

For an equation $F(z) = 0$ involving exponentials or trigonometric functions:

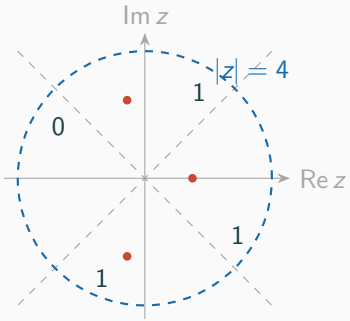
1. choose a region with a simple boundary;
2. split $F = f + g$ so one term dominates;
3. verify strict domination everywhere on the contour;
4. count the zeros of the simpler comparison function.

Limitation

The theorem counts roots; it does not accurately locate each root. Numerical refinement is a separate step.

Illustrative example: count first, locate second

$$F(z) = z - \frac{\sin z}{2} - 1, \qquad |z| < 4.$$



Count and interpretation

Argument-principle counts show that three roots lie in the disk. Numerical refinement then gives:

$$z_1 \approx 1.499,$$

$$z_{2,3} \approx -0.559 \pm 2.458\mathrm{i}.$$

The sector counts $0, 1, 1, 1$ indicate where to start the numerical searches; counting and locating remain distinct steps.

Summary: counting zeros

Method	Best when	What it returns
argument principle	f'/f or phase is accessible	$N - P$
phase unwrapping	$f(C)$ can be sampled numerically	winding number
Rouché	one term dominates on C	number of zeros
direct/numerical roots	locations are needed	approximate root positions

Main message

Counting and locating roots are different tasks; choose the theorem to match the question.

Practice

Pause and predict

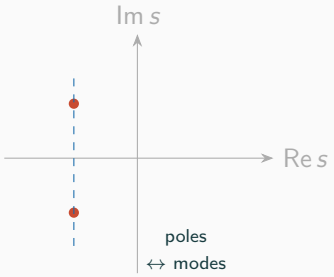
Decide first whether the question asks for a *count* or a *location*; then choose the theorem.

1. If f has three zeros and one simple pole inside C, what is $\dfrac{1}{2\pi \mathrm{i}} \oint_C f'(z)/f(z)\,\mathrm{d}z$?
2. Explain geometrically what it means if $f(C)$ winds twice counterclockwise around the origin.
3. Use Rouché's theorem to count the zeros of $z^4 + 2z + 1$ in $|z| < 2$.
4. Why must the phase be unwrapped when the argument principle is implemented numerically?

Special Functions and Engineering Interpretation

Key points

- Gamma function: extend factorials by an integral representation
- Analytic continuation: understand where poles appear
- Reflection formula: reconnect the keyhole contour with a special-function identity
- Transform viewpoint: interpret poles as physical modes
- Separate what is introduced here from what later lectures develop systematically



Why does the gamma function belong in this lecture?

Earlier, the keyhole contour produced

$$\int_0^\infty \frac{x^{\alpha-1}}{1+x}\,\mathrm{d}x = \frac{\pi}{\sin(\pi\alpha)}, \qquad 0 < \operatorname{Re}\alpha < 1.$$

The same real integral also appears naturally in the identity

$$\Gamma(\alpha)\Gamma(1-\alpha) = \int_0^\infty \frac{x^{\alpha-1}}{1+x}\,\mathrm{d}x.$$

Therefore the contour result is not an isolated trick: it becomes a structural identity for a special function.

What students should take from this section

We use Γ as one example showing how contour methods create reusable formulas. A systematic study of transforms comes later.

The gamma function extends the factorial

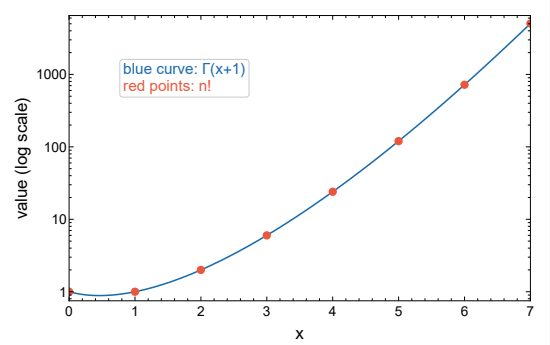
For $\operatorname{Re} z > 0$,

$$\Gamma(z) = \int_0^\infty t^{z-1} \mathrm{e}^{-t}\, \mathrm{d}t.$$

Integration by parts gives

$$\Gamma(z+1) = z\Gamma(z),$$

so, for integers $n \geq 0$, $\Gamma(n+1) = n!$.



Geometric meaning

The continuous curve interpolates the discrete factorial values.

Why the defining integral converges

In

$$\Gamma(z) = \int_0^\infty t^{z-1} \mathrm{e}^{-t}\, \mathrm{d}t,$$

the two endpoints impose different conditions:

- near $t = 0$, $|t^{z-1}| = t^{\operatorname{Re} z - 1}$ is integrable iff $\operatorname{Re} z > 0$;
- as $t \to \infty$, exponential decay dominates every algebraic power.

The recurrence $\Gamma(z) = \Gamma(z+1)/z$ then continues the function across successive left half-planes, except at $0, -1, -2, \ldots$.

Poles of the gamma function

Repeatedly applying the recurrence gives

$$\Gamma(z) = \frac{\Gamma(z+n+1)}{z(z+1)\cdots(z+n)}.$$

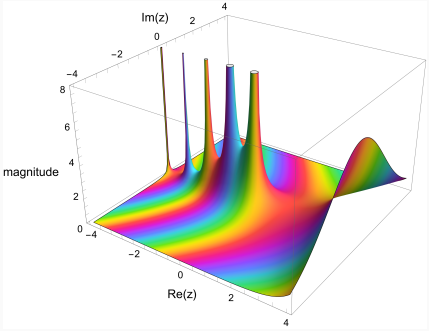
Hence Γ has simple poles at $z = -n$, $n = 0, 1, 2, \ldots$, with

$$\operatorname{Res}(\Gamma; -n) = \frac{(-1)^n}{n!}.$$

Application viewpoint

These poles organize asymptotic coefficients and appear naturally in transform inversion and special-function identities.

The poles become visible on the complex surface



Read the surface

- Height represents $|\Gamma(z)|$.
- Color records complex phase.
- Vertical spikes occur at $0, -1, -2, \ldots$.

Away from the spikes, analytic continuation produces a smooth complex surface.

Reflection links a real integral to residues

The beta-type integral gives, for $0 < \operatorname{Re} z < 1$,

$$\Gamma(z)\Gamma(1-z) = \int_0^\infty \frac{t^{z-1}}{1+t}\,\mathrm{d}t.$$

The keyhole-contour result therefore yields Euler's reflection formula

$$\boxed{\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}}.$$

Branch contours produce identities for special functions, not only isolated definite integrals.

A first look at transform methods

Many engineering models contain the kernel e^{-st}:

$$F(s) = \int_0^\infty f(t)\mathrm{e}^{-st}\,\mathrm{d}t.$$

Complex analysis explains

- why F is analytic in a convergence half-plane;
- how poles encode exponential modes;
- why residues recover modal contributions;
- how contour deformation supports inversion.

L09 introduces transform language; L10 develops Laplace theory.

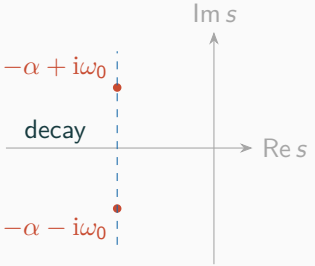
Poles predict engineering modes

Consider

$$H(s) = \frac{1}{(s+\alpha)^2 + \omega_0^2}, \qquad \alpha > 0.$$

The poles $s = -\alpha \pm \mathrm{i}\omega_0$ produce

$$h(t) = \frac{1}{\omega_0} \mathrm{e}^{-\alpha t} \sin(\omega_0 t).$$



Residues are modal amplitudes

If a transform has distinct poles s_k and can be inverted by a leftward contour closure, then formally

$$f(t) = \sum_k \operatorname{Res}\big(F(s)\mathrm{e}^{st}; s_k\big)\,.$$

Each pole contributes a mode:

Pole pattern	Time-domain behavior
$s = -\alpha$	$\mathrm{e}^{-\alpha t}$
$s = -\alpha \pm \mathrm{i}\omega$	damped oscillation
repeated pole of order m	polynomial t^{m-1} times an exponential

L12 develops the inversion conditions and computation systematically.

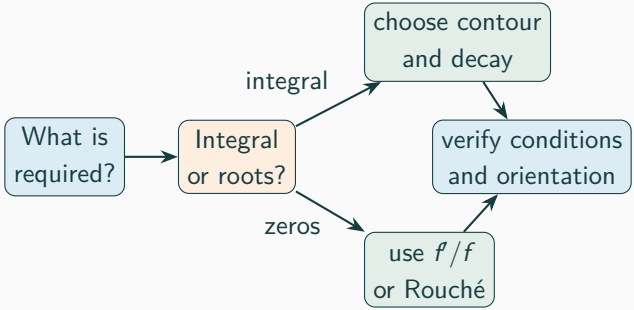
Summary: special functions and engineering interpretation

- The gamma function extends factorials and acquires simple poles through analytic continuation.
- The keyhole-contour integral reappears as Euler's reflection formula.
- In transform problems, pole locations predict decay and oscillation, while residues determine modal amplitudes.

Main message

Complex singularities are not only algebraic objects: they organize identities, modes, and engineering behavior.

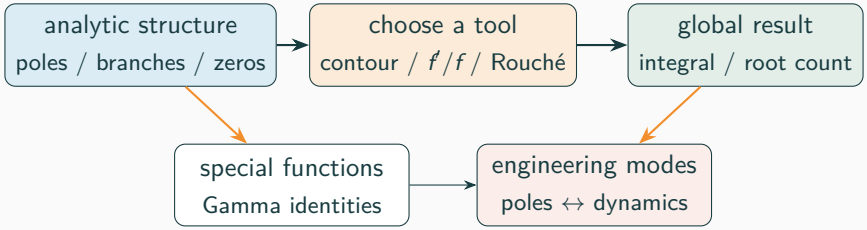
A unified decision workflow



The theorem is selected only after the geometry and analytic conditions are clear.

What we have learned & Concept map

- Residues become useful only after the contour is designed to match the problem.
- Unit-circle, semicircular, indented, and keyhole contours handle different analytic structures.
- The argument principle turns zeros minus poles into a winding number; Rouché counts zeros by comparison.
- Gamma-function identities and transform modes show how local singularities encode global information.



Practice: connect formulas with meaning

1. Use $\Gamma(z+1) = z\Gamma(z)$ to compute $\Gamma(5)$ and relate it to a factorial.
2. From the recurrence, explain why $0, -1, -2, \ldots$ are poles of Γ.
3. For poles $s = -2 \pm 3\mathrm{i}$, predict the qualitative time-domain behavior without carrying out a full inverse transform.
4. Explain in one sentence how the beta-type keyhole integral reappears in Euler's reflection formula.

Looking ahead: from complex analysis to transforms

Topic	Primary lecture	Role here
hydrodynamics and conformal transfer	L07	bridge only
real integrals and root counting	L08	primary
Fourier transform	L09	systematic
Laplace definition and ROC	L10	systematic
inversion and modal residues	L12	systematic
control poles and stability	L15	application

Next $\gg$ Lecture 09 — Integral Transforms and Fourier Transform