L09 — Integral Transforms and Fourier Transform

Engineering Mathematics I

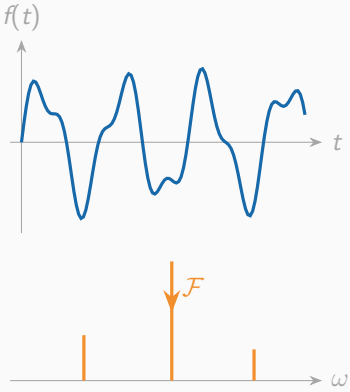
Profs. Tianyu DONG & Jiawei WANG

Fall 2026

School of Electrical Engineering, Xi'an Jiaotong University

Road map

- Integral transforms and kernel representations
- Fourier series for periodic signals
- Fourier transform for nonperiodic signals
- Properties and engineering applications



A signal viewed through its frequencies

Consider a signal containing a slow oscillation and a faster disturbance:

$$x(t) = \cos t + 0.3\cos(8t).$$

Time description	Frequency description
Both oscillations contribute to each value of $x(t)$.	The components have angular frequencies 1 and 8 rad/s.

A filter can retain the slow component and suppress the faster one. Fourier analysis supplies this frequency description for more general signals.

Why transform a problem?

A transform changes representation while preserving information:

$$f(t) \quad \longleftrightarrow \quad F(\omega).$$

Time or space domain	**Frequency domain**
Localization, causality, discontinuities, differential equations.	Oscillation, bandwidth, filtering, resonance, algebraic multiplication.

The best representation is the one in which the operation of interest becomes simplest.

Integral Transforms and Kernel Representations

A general integral transform

An integral transform has the form

$$F(s) = \mathcal{T}\{f\}(s) = \int_a^b K(s,t) f(t)\,\mathrm{d}t,$$

where $K(s,t)$ is the **kernel**.

Transform	Kernel	Typical purpose
Fourier	$\mathrm{e}^{-\mathrm{i}\omega t}$	frequency analysis
Laplace	e^{-st}	causal dynamics and growth
sine/cosine	$\sin \omega t$, $\cos \omega t$	boundary problems

The kernel defines what the transform measures

The quantity

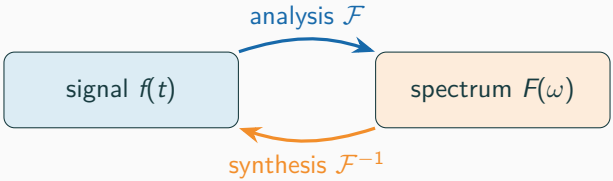
$$F(s) = \int_a^b f(t)K(s,t)\,\mathrm{d}t$$

is a weighted comparison between the signal and a family of kernels.



Fourier kernels measure oscillation; Laplace kernels measure oscillation together with exponential growth or decay.

Analysis and synthesis



An inverse formula is not optional: without it, the transform is only a feature extractor, not an equivalent representation.

Fourier Series for Periodic Signals

Key points

- Harmonics and orthogonality
- Sine–cosine and complex coefficients
- Symmetry and worked examples
- Convergence and half-range extensions
- Average power and approximation

Periodic representation

$$f(t) \sim \sum_{n\in\mathbb{Z}} c_n e^{in\omega_0 t}.$$

One coefficient for each discrete harmonic.

Periodic signals and harmonic building blocks

For a signal with period T, set $\omega_0 = 2\pi/T$. The functions

$$1, \quad \cos(n\omega_0 t), \quad \sin(n\omega_0 t), \qquad n = 1, 2, \ldots$$

all repeat after time T. We seek a representation

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)].$$

The next step

Orthogonality lets us calculate each coefficient separately. The symbol $\sim$ denotes a Fourier-series representation until convergence is specified.

Orthogonality extracts one harmonic

For positive integers m, n, averaging over one full period gives

$$\frac{2}{T}\int_{-T/2}^{T/2} \cos(m\omega_0 t)\cos(n\omega_0 t)\,\mathrm{d}t = \begin{cases} 1, & m = n, \\ 0, & m \neq n. \end{cases}$$

The same relation holds for two sine terms. Sine–cosine products and the mean of each sine or cosine are zero.

Coefficient extraction

Multiply the series by $\cos(n\omega_0 t)$ and integrate. Only the term containing a_n survives.

Fourier coefficients in sine–cosine form

Integrate over any complete period $[t_0, t_0 + T]$:

$$a_0 = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\,\mathrm{d}t, \qquad \text{mean value} = \frac{a_0}{2}.$$

$$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos(n\omega_0 t)\,\mathrm{d}t,$$

$$b_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin(n\omega_0 t)\,\mathrm{d}t, \qquad n \geq 1.$$

Normalization check

A constant signal $f(t) = C$ must give $a_0 = 2C$ and $a_n = b_n = 0$ for $n \geq 1$.

Symmetry removes half the work

Choose one period $[-L, L]$, so $T = 2L$ and $\omega_0 = \pi/L$:

Even f

$b_n = 0$; only cosine terms remain.

$$a_n = \frac{2}{L}\int_0^L f(t)\cos\frac{n\pi t}{L}\,\mathrm{d}t.$$

Odd f

$a_n = 0$; only sine terms remain.

$$b_n = \frac{2}{L}\int_0^L f(t)\sin\frac{n\pi t}{L}\,\mathrm{d}t.$$

Odd symmetry and half-wave symmetry

These symmetries answer different coefficient questions:

- $f(-t) = -f(t)$ removes the mean and cosine terms.
- $f(t + T/2) = -f(t)$ removes all even-numbered harmonics.

For the square wave

Both conditions hold. Its series therefore contains only $\sin(\omega_0 t), \sin(3\omega_0 t), \sin(5\omega_0 t), \ldots$.

An odd signal need not have only odd-numbered harmonics. For example, $\sin t + \sin(2t)$ is odd and contains an even harmonic.

Worked example: the square wave

Let $f(t)$ be 2π-periodic and

$$f(t) = \begin{cases} -1, & -\pi < t < 0, \\ 1, & 0 < t < \pi. \end{cases}$$

Sine coefficients

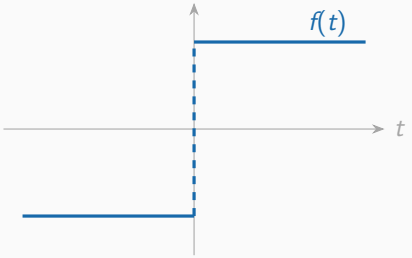
Odd symmetry removes the mean and cosine terms:

$$b_n = \frac{2}{\pi} \int_0^{\pi} \sin(nt)\,\mathrm{d}t = \begin{cases} 4/(n\pi), & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases}$$

Hence

$$\boxed{f(t) \sim \frac{4}{\pi}\left(\sin t + \frac{\sin 3t}{3} + \frac{\sin 5t}{5} + \cdots\right).}$$

The square wave has an odd-harmonic line spectrum





Half-wave symmetry predicts the missing even harmonics. The plotted heights are $|b_n| = 4/(n\pi)$ for odd n.

What does a Fourier series converge to?

For a periodic, piecewise C^1 signal with finitely many pieces per period,

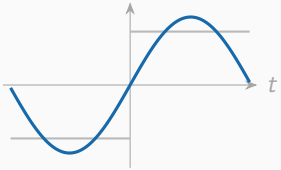
$$S_f(t) = \frac{f(t^-) + f(t^+)}{2}.$$

Therefore

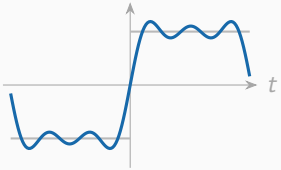
- at a continuity point, $S_f(t) = f(t)$;
- at a jump, the series converges to the midpoint;
- endpoint values follow the periodic extension.

For the square wave, the series equals 0 at $t = 0, \pm\pi$. For the periodic sawtooth introduced next, it equals 0 at $t = \pm\pi$.

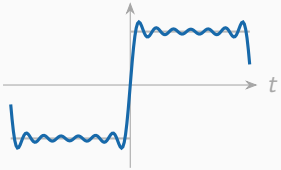
Partial sums reveal the Gibbs phenomenon



1 term



3 terms



7 terms

- away from jumps, the approximation improves as N increases;
- near a jump, the ringing region narrows but the limiting overshoot is about 9% of the jump;
- at the jump, the partial sums converge to the midpoint value.

Worked example: a sawtooth signal

Worked example

Let $f(t) = t$ on $(-\pi, \pi)$ and extend it periodically. Since f is odd,

$$b_n = \frac{2}{\pi} \int_0^{\pi} t \sin(nt)\, \mathrm{d}t = \frac{2(-1)^{n+1}}{n}.$$

Hence

$$\boxed{t \sim 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nt)}, \qquad -\pi < t < \pi.$$

Compared with the square wave, every harmonic is present and its amplitude still decays like $1/n$ because of the jump in the periodic extension.

Amplitude–phase form of each harmonic

Combine the sine and cosine terms as

$$a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) = A_n \cos(n\omega_0 t - \phi_n),$$

where

$$A_n = \sqrt{a_n^2 + b_n^2}, \qquad \phi_n = \operatorname{atan2}(b_n, a_n).$$

Engineering reading

A_n is the harmonic amplitude. In the usual $\cos(n\omega_0 t + \theta_n)$ convention, $\theta_n = -\phi_n$. If $A_n = 0$, its phase is undefined.

Complex Fourier series and coefficient conversion

Euler's formula combines the sine and cosine terms:

$$f(t) \sim \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{\mathrm{i}n\omega_0 t}, \qquad c_n = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\mathrm{e}^{-\mathrm{i}n\omega_0 t}\,\mathrm{d}t.$$

$$c_0 = \frac{a_0}{2}, \qquad c_n = \frac{a_n - \mathrm{i}b_n}{2}, \qquad c_{-n} = \frac{a_n + \mathrm{i}b_n}{2} \quad (n \geq 1).$$

For real signals

$c_{-n} = \overline{c_n}$ and $A_n = 2|c_n|$ for $n \geq 1$. Positive and negative frequencies together represent a real oscillation.

Deriving one Fourier coefficient

Start from a finite harmonic expansion

$$f(t) = \sum_{m} c_m \mathrm{e}^{\mathrm{i} m \omega_0 t}.$$

Multiply by $\mathrm{e}^{-\mathrm{i} n \omega_0 t}$ and average over one period:

$$\frac{1}{T} \int_{t_0}^{t_0+T} f(t) \mathrm{e}^{-\mathrm{i} n \omega_0 t} \, \mathrm{d}t = \sum_{m} c_m \frac{1}{T} \int_{t_0}^{t_0+T} \mathrm{e}^{\mathrm{i}(m-n)\omega_0 t} \, \mathrm{d}t.$$

Orthogonality removes every term except $m = n$, yielding the coefficient formula.

Half-range data and the extension choice

Suppose f is given only on $0 < t < L$. We choose an extension on $(-L, L)$, then repeat it with period $2L$.

Even extension

$f_e(-t) = f_e(t)$ gives a cosine series. Cosine basis functions have zero derivative at 0 and L.

Odd extension

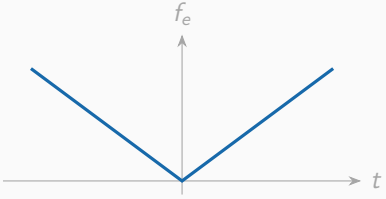
$f_o(-t) = -f_o(t)$ gives a sine series. Sine basis functions vanish at 0 and L.

These are useful bases for homogeneous Neumann and Dirichlet conditions, respectively. Endpoint convergence still depends on the extension.

Even and odd extensions lead to different spectra

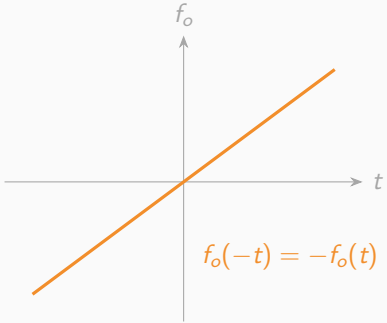
For the same half-range data $f(t) = t$ on $0 < t < L$, two extensions are possible:

Even extension: cosine series



$$f_e(-t) = f_e(t)$$

Odd extension: sine series



$$f_o(-t) = -f_o(t)$$

Choice, not convention: the extension determines endpoint behavior and must follow the model's boundary conditions.

Worked example: two expansions of the same half-range data

For $f(t) = t$ on $0 < t < L$, the same coefficient formulas give

$$\text{Sine:} \qquad t = \frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi t}{L},$$

$$\text{Cosine:} \qquad t = \frac{L}{2} - \frac{4L}{\pi^2} \sum_{\substack{n \geq 1 \\ n \text{ odd}}} \frac{1}{n^2} \cos \frac{n\pi t}{L}.$$

Both represent t in the open interval. At $t = L$, the sine series gives 0 and the cosine series gives L.

Why the endpoints differ

The odd periodic extension has a jump at L. The even periodic extension is continuous there.

Endpoint matching controls coefficient decay

The previous examples show two different rates:

Periodic extension	Least smooth feature	Decay
Square wave / sawtooth	jump in the signal	$O(n^{-1})$
Even extension of t	jump in the slope	$O(n^{-2})$

Integration by parts explains the pattern. Each additional integration supplies a factor $1/n$ when the derivatives and boundary terms allow it.

The boundary matters

Smoothness must be checked across the periodic join as well as inside the interval.

Parseval: average power in two representations

For a square-integrable periodic signal,

$$\frac{1}{T}\int_{t_0}^{t_0+T} |f(t)|^2 \,\mathrm{d}t = \sum_{n=-\infty}^{\infty} |c_n|^2.$$

For real f, the equivalent sine–cosine formula is

$$\frac{1}{T}\int_{t_0}^{t_0+T} |f(t)|^2 \,\mathrm{d}t = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

Interpretation

The left side is average power. Each term on the right is a harmonic contribution. The integral over one period is T times the average power.

Worked example: Parseval evaluates a series

Worked example

Apply the real Parseval identity to the sawtooth expansion $f(t) = t$:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} t^2\,\mathrm{d}t = \sum_{n=1}^{\infty} b_n^2 = 4\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

Since the left side is $2\pi^2/3$,

$$\boxed{\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}}.$$

A spectral energy identity has produced a nontrivial numerical sum.

Fourier partial sums are best mean-square approximations

Among all trigonometric polynomials of degree at most N, the Fourier partial sum S_N minimizes

$$E_N = \frac{1}{T}\int_{t_0}^{t_0+T} |f(t) - S_N(t)|^2 \, \mathrm{d}t.$$

Orthogonality gives the energy decomposition

$$\frac{1}{T}\int_{t_0}^{t_0+T} |f|^2 \, \mathrm{d}t = \sum_{|n|\le N} |c_n|^2 + E_N,$$

so truncating small coefficients has a precise approximation-error meaning.

Engineering interpretation

S_N is optimal when the allowed frequencies are fixed to $|n| \le N$. Its error is the total power in the omitted coefficients.

Fourier-series method checklist

1. Identify the fundamental period and choose a convenient integration interval.
2. Test even, odd, and half-wave symmetry before calculating coefficients.
3. Select complex, sine–cosine, or amplitude–phase form for the task.
4. Compute representative coefficients and check their decay against signal smoothness.
5. State the convergence value at jumps and endpoints.
6. Use Parseval or truncation error to check the approximation quantitatively.

Common error

A correct coefficient formula can still represent the wrong periodic extension if the interval and boundary interpretation are not checked.

Practice

1. For $f(t) = t$ on $(-\pi, \pi)$ with periodic extension, find T and ω_0.
2. Which coefficients vanish? Does odd symmetry remove even harmonics?
3. Find the series value at $t = \pi$ using the periodic one-sided limits.
4. For $g(t) = |t|$ on $(-\pi, \pi)$, choose a sine or cosine expansion and predict the coefficient decay.

Answer check

For f, $T = 2\pi$, $\omega_0 = 1$, and $a_0 = a_n = 0$. Even harmonics remain, and the sum at π is 0. For g, use a cosine series with $O(n^{-2})$ coefficients.

Summary: Fourier series

Question	Answer
Which signals?	Periodic signals with period T and $\omega_0 = 2\pi/T$.
Coefficients?	Integrate against one harmonic over a complete period. Symmetry can eliminate terms.
Spectrum?	Discrete frequencies $n\omega_0$, with coefficients c_n or real amplitudes A_n.
Reconstruction?	For piecewise C^1 signals, the value at continuity points and the midpoint at jumps.
Accuracy?	Parseval gives average power. The omitted coefficients determine mean-square error.

Main message

The periodic extension determines both the endpoint values and the harmonic content.

Fourier Transform for Nonperiodic Signals

Key points

- The limit of a long-period signal
- Transform and inverse conventions
- Existence and reconstruction
- Pulse, exponential, and Gaussian spectra
- Half-line sine and cosine transforms

Continuous representation

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \mathrm{e}^{\mathrm{i}\omega t} \, \mathrm{d}\omega.$$

A continuous spectral density, under inversion assumptions.

From discrete harmonics to a continuous spectrum

As the period T grows,

$$\omega_n = n\frac{2\pi}{T}, \qquad \Delta\omega = \frac{2\pi}{T} \longrightarrow 0.$$

The Fourier-series sum approaches a Fourier integral: discrete spectral lines merge into a continuous spectrum.



How the Fourier integral emerges

Define $F(\omega) = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t$. For a truncated signal repeated every T,

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\mathrm{e}^{-\mathrm{i}\omega_n t}\,\mathrm{d}t \approx \frac{\Delta\omega}{2\pi}F(\omega_n).$$

Substitution into the Fourier series gives the Riemann sum

$$f(t) \approx \frac{1}{2\pi}\sum_n F(\omega_n)\mathrm{e}^{\mathrm{i}\omega_n t}\Delta\omega.$$

When the limiting process is valid, this Riemann sum gives the inverse Fourier integral. $F(\omega)$ is a spectral density, whereas c_n is a discrete coefficient.

Fourier transform pair and convention

We use

$$\boxed{F(\omega) = \int_{-\infty}^{\infty} f(t) \mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t},$$

$$\boxed{f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega}.$$

Other conventions distribute the factor 2π differently. Always state the convention before using a transform table.

Existence: useful sufficient conditions

A simple sufficient condition is absolute integrability:

$$\int_{-\infty}^{\infty} |f(t)|\,\mathrm{d}t < \infty.$$

Then $F(\omega)$ is bounded and continuous. More general transforms exist in an L^2 or distributional sense.

Course boundary

A nonzero periodic signal is usually not absolutely integrable on the whole real line. We use Fourier series for it. Generalized transforms of periodic signals require impulses and are outside the core calculation here.

Fourier inversion and jump values

For $f \in L^1(\mathbb{R})$ that is locally piecewise C^1, interpret the inverse as a symmetric limit $\lim_{R\to\infty} \int_{-R}^{R}$: inversion gives

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega = \frac{f(t^-)+f(t^+)}{2}.$$

Thus the same midpoint rule appears for both Fourier series and Fourier integrals.

Interpretation

The transform reconstructs the signal at continuity points and the average of one-sided limits at a jump.

Worked example: a rectangular pulse

Let

$$f(t) = \begin{cases} 1, & |t| < W/2, \\ 0, & |t| > W/2. \end{cases}$$

Then

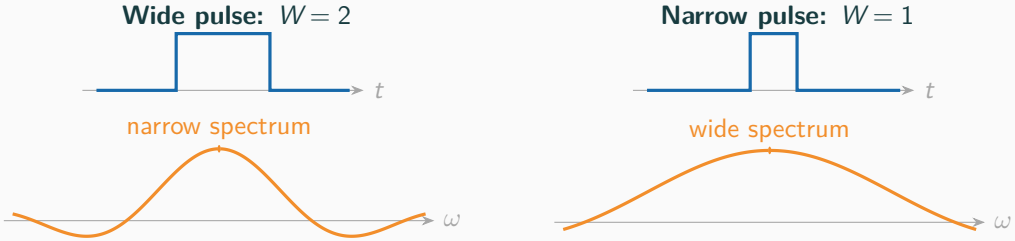
$$F(\omega) = \int_{-W/2}^{W/2} \mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t = W \operatorname{sinc}\left(\frac{\omega W}{2}\right),$$

where $\operatorname{sinc} x = \sin x / x$ for $x \neq 0$ and $\operatorname{sinc}(0) = 1$. Thus $F(0) = W$.

Time–frequency tradeoff

A wider pulse has a narrower main spectral lobe; a shorter pulse spreads energy across a wider frequency range.

Pulse width controls spectral width



For the normalized spectrum $F(\omega)/W$, the first zero moves from π to 2π when the pulse width is halved.

Reading the rectangular-pulse spectrum

For $F(\omega) = W\,\mathrm{sinc}(\omega W/2)$:

- $F(0) = W$ equals the pulse area;
- zeros occur at $\omega = \pm 2\pi/W, \pm 4\pi/W, \ldots$;
- the main-lobe width is inversely proportional to W;
- alternating side lobes record the sharp pulse edges.

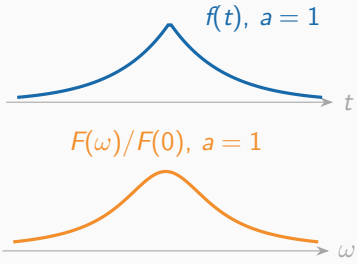
The spectrum is not merely a picture: its zeros and width can be predicted directly from the time-domain duration.

Worked example: a decaying two-sided exponential

Let $f(t) = \mathrm{e}^{-a|t|}$ with $a > 0$. Since f is even,

$$F(\omega) = 2\int_0^\infty \mathrm{e}^{-at}\cos(\omega t)\,\mathrm{d}t = \boxed{\frac{2a}{a^2+\omega^2}}.$$

- larger a gives faster time decay and a broader spectrum;
- algebraic spectral tails reflect the cusp at $t = 0$;
- $F(0) = 2/a$ equals the total area under f.

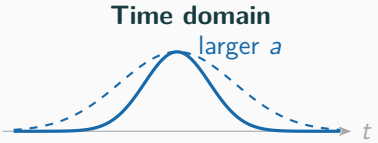


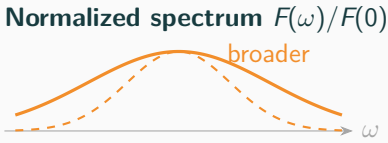
Worked example: the Gaussian is self-similar

For $a > 0$,

$$f(t) = \mathrm{e}^{-at^2} \quad \Longrightarrow \quad F(\omega) = \sqrt{\frac{\pi}{a}}\mathrm{e}^{-\omega^2/(4a)}.$$

A Gaussian transforms into another Gaussian. Increasing a narrows the signal in time but broadens its spectrum.





Half-line transforms and their full-line extensions

For f specified on $t > 0$, define, for $\omega \geq 0$,

$$F_c(\omega) = \int_0^\infty f(t)\cos(\omega t)\,\mathrm{d}t, \qquad F_s(\omega) = \int_0^\infty f(t)\sin(\omega t)\,\mathrm{d}t.$$

If f_e and f_o are its even and odd extensions, then

$$\mathcal{F}\{f_e\}(\omega) = 2F_c(\omega), \qquad \mathcal{F}\{f_o\}(\omega) = -2\mathrm{i}F_s(\omega) \quad (\omega \geq 0).$$

Under the corresponding inversion assumptions, for $t > 0$,

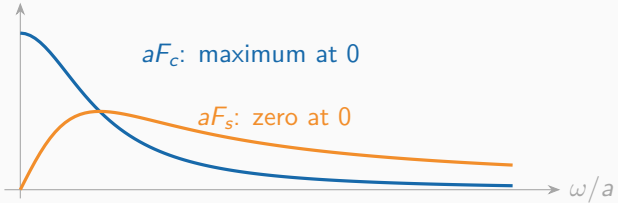
$$f(t) = \frac{2}{\pi}\int_0^\infty F_c(\omega)\cos(\omega t)\,\mathrm{d}\omega,$$

with the analogous formula using F_s and $\sin(\omega t)$. At a jump, each inverse gives the midpoint of the one-sided limits.

Worked example: half-line exponential data

Let $f(t) = \mathrm{e}^{-at}$ on $t > 0$, with $a > 0$. Direct integration gives

$$F_c(\omega) = \frac{a}{a^2 + \omega^2}, \qquad F_s(\omega) = \frac{\omega}{a^2 + \omega^2}.$$



The plot compares aF_c and aF_s against ω/a (common vertical display scale). The even full-line transform is real, and the odd one is imaginary.

Practice

1. A unit-height pulse has width W. Find $F(0)$ and its first positive spectral zero.
2. Repeat for a pulse of width $W/2$. Compare the peak and main-lobe width.
3. For $f(t) = \mathrm{e}^{-a|t|}$, verify $F(0)$ by calculating the time-domain area.

Answer check

The width-W pulse gives $F(0) = W$ and first zero $2\pi/W$. Halving the width halves the peak and doubles the main-lobe width. For the exponential, $F(0) = 2/a$.

Summary: the Fourier transform

Question	Answer
Which signals?	L^1 signals have an ordinary transform. Finite-energy signals also have an L^2 transform.
Calculation?	$F(\omega) = \int_{-\infty}^{\infty} f(t) \mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t$.
Spectrum?	A continuous density of frequency components. Time compression broadens the spectrum.
Reconstruction?	Under inversion assumptions, the original value at continuity points and the midpoint at jumps.
Convention?	Angular frequency ω; the factor $1/(2\pi)$ belongs to the inverse transform.

Main message

The normalization, convergence assumptions, and meaning of the spectral variable must stay consistent.

Properties and Engineering Applications

Key points

- Linearity, delay, modulation, and scaling
- Differentiation and differential equations
- Convolution and filtering
- Energy and correlation

The system relation

$$y = x * h, \qquad Y = HX.$$

A frequency response weights each spectral component.

Linearity and a transform calculation without integration

Worked example

Suppose $p(t) \longleftrightarrow P(\omega)$ and $q(t) \longleftrightarrow Q(\omega)$. Then

$$\mathcal{F}\{\alpha p + \beta q\} = \alpha P + \beta Q.$$

For example,

$$f(t) = 2\mathrm{e}^{-|t|} - \mathrm{e}^{-2|t|}$$

has transform

$$F(\omega) = \frac{4}{1+\omega^2} - \frac{4}{4+\omega^2}.$$

Check at zero: $F(0) = 3 = \int_{-\infty}^{\infty} f(t)\,\mathrm{d}t$.

Time shift and modulation

If $f(t) \longleftrightarrow F(\omega)$, then

$$f(t - t_0) \longleftrightarrow \mathrm{e}^{-\mathrm{i}\omega t_0} F(\omega),$$

$$\mathrm{e}^{\mathrm{i}\omega_0 t} f(t) \longleftrightarrow F(\omega - \omega_0).$$

A time delay changes spectral phase but not magnitude.

Modulation translates a spectrum to a carrier frequency.

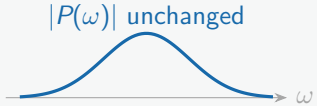
Worked example: delay versus modulation

Let $p(t)$ be a rectangular pulse with transform $P(\omega)$.

Delay

$$p(t - t_0) \longleftrightarrow \mathrm{e}^{-\mathrm{i}\omega t_0} P(\omega).$$

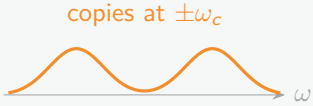
The magnitude is unchanged; only a linear phase is added.



Modulation

$$p(t)\cos(\omega_c t) \longleftrightarrow \tfrac{1}{2}\big[P(\omega - \omega_c) + P(\omega + \omega_c)\big].$$

Two translated spectral copies appear around $\pm\omega_c$.



Scaling expresses the uncertainty tradeoff

For nonzero a,

$$f(at) \longleftrightarrow \frac{1}{|a|} F\left(\frac{\omega}{a}\right).$$

Therefore

- compressing a signal in time expands its spectrum;
- stretching a signal in time compresses its spectrum;
- the amplitude factor preserves the integral relation.

This property explains the rectangular-pulse and Gaussian examples without recomputing their integrals.

Differentiation in the transform domain

Suppose $f(t), f'(t) \in L^1(\mathbb{R})$, f is absolutely continuous, and $f(t) \to 0$ at both infinities. Integration by parts gives

$$\mathcal{F}\{f'(t)\} = \left[f(t)\mathrm{e}^{-\mathrm{i}\omega t}\right]_{-\infty}^{\infty} + \mathrm{i}\omega \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t = \mathrm{i}\omega F(\omega).$$

Repeating the argument under analogous assumptions gives

$$\mathcal{F}\{f^{(n)}\} = (\mathrm{i}\omega)^n F.$$

If also $tf(t) \in L^1$, then

$$\mathcal{F}\{tf(t)\} = \mathrm{i}\frac{\mathrm{d}F(\omega)}{\mathrm{d}\omega}.$$

Differential equations can therefore become algebraic equationsiers, which is why transforms are effective for linear differential equations.

Worked example: a stable first-order system

Worked example

Consider $y'(t) + ay(t) = x(t)$ with $a > 0$ and a decaying input. Choose the response with no free transient from the remote past.

$$(a + \mathrm{i}\omega)\,Y(\omega) = X(\omega), \qquad H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{a + \mathrm{i}\omega}.$$

Thus $Y = HX$ and

$$|H(\omega)| = \frac{1}{\sqrt{a^2 + \omega^2}}.$$

High frequencies receive less gain than low frequencies. Next we connect this multiplication to a time-domain operation. Initial conditions specified at a finite time will be treated using Laplace transforms.

Convolution becomes multiplication

Define

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau) g(t - \tau)\, \mathrm{d}\tau.$$

For absolutely integrable f and g,

$$\boxed{\mathcal{F}\{f * g\} = F(\omega) G(\omega)}.$$

When the relevant integrals exist, our convention also gives

$$\mathcal{F}\{fg\} = \frac{1}{2\pi}(F * G).$$

For a linear time-invariant (LTI) system with impulse response h, the output is $y = x * h$, so $Y = XH$.

Worked example: the same system in the time domain

Worked example

Define $h(t) = \mathrm{e}^{-at}u(t)$, where $a > 0$ and $u(t) = 1$ for $t > 0$, 0 for $t < 0$. Direct integration gives

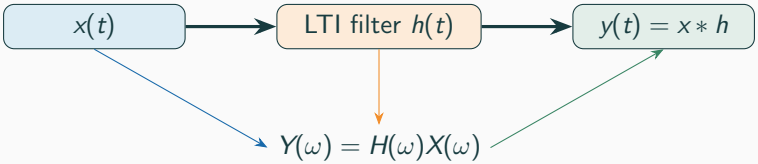
$$H(\omega) = \int_0^\infty \mathrm{e}^{-(a+\mathrm{i}\omega)t}\,\mathrm{d}t = \frac{1}{a+\mathrm{i}\omega}.$$

The response $Y = HX$ therefore has the time-domain form

$$y(t) = (x * h)(t) = \int_{-\infty}^{t} x(\tau)\mathrm{e}^{-a(t-\tau)}\,\mathrm{d}\tau.$$

Differentiation verifies $y'(t) = x(t) - ay(t)$. Since $h(t) = 0$ for $t < 0$, only past input values contribute.

Filtering in one picture



Convolution in time is multiplication in frequency, so a filter acts by weighting each frequency component.

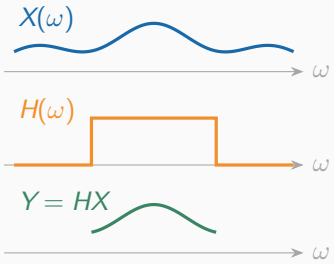
Worked example: ideal low-pass filtering

Suppose

$$H(\omega) = \begin{cases} 1, & |\omega| \le \omega_c, \\ 0, & |\omega| > \omega_c. \end{cases}$$

Then $Y(\omega) = H(\omega)X(\omega)$ retains only the passband, and

$$h(t) = \frac{\sin(\omega_c t)}{\pi t}, \quad h(0) = \frac{\omega_c}{\pi}.$$



The ideal filter is noncausal: h extends to negative time. The transform pair uses a limiting integral; the plots are schematic.

Parseval's theorem for the transform

For square-integrable signals,

$$\int_{-\infty}^{\infty} |f(t)|^2 \, \mathrm{d}t = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 \, \mathrm{d}\omega.$$

Engineering meaning

Total signal energy is independent of whether it is computed in time or frequency.

This identity supports energy checks, spectral-density interpretation, and approximation-error estimates.

Checkpoint: energy of a rectangular pulse

A unit-height pulse of width W has energy

$$E = \int_{-\infty}^{\infty} |p(t)|^2 \, \mathrm{d}t = W.$$

Parseval requires the same result from its spectrum:

$$W = \frac{1}{2\pi} \int_{-\infty}^{\infty} W^2 \operatorname{sinc}^2(\omega W/2) \, \mathrm{d}\omega.$$

With $v = \omega W/2$, this is equivalent to

$$\int_{-\infty}^{\infty} \operatorname{sinc}^2 v \, \mathrm{d}v = \pi.$$

Halving the pulse width halves its energy and doubles its main-lobe width.

Correlation and the sign of a delay

For finite-energy signals, use the convention

$$r_{xy}(\tau) = \int_{-\infty}^{\infty} x(t)\overline{y(t-\tau)}\,\mathrm{d}t, \qquad R_{xy}(\omega) = X(\omega)\overline{Y(\omega)}.$$

For $y(t) = x(t-d)$, substitution gives

$$r_{xy}(\tau) = r_{xx}(\tau + d).$$

A delayed copy with $d > 0$ therefore aligns at $\tau = -d$ under this convention.

Autocorrelation

$R_{xx} = |X|^2$ and $r_{xx}(0) = \int |x(t)|^2\,\mathrm{d}t$. Correlation connects alignment in time with spectral energy.

Fourier-transform property reference

Under the assumptions stated with each property, $f \longleftrightarrow F$:

Time domain operation	Frequency-domain effect	Typical use
$f(t - t_0)$	$\mathrm{e}^{-\mathrm{i}\omega t_0} F(\omega)$	timing and propagation
$\mathrm{e}^{\mathrm{i}\omega_c t} f(t)$	$F(\omega - \omega_c)$	communication carriers
$f(at)$, $a \neq 0$	$\lvert a \rvert^{-1} F(\omega/a)$	bandwidth estimates
$f'(t)$	$\mathrm{i}\omega F(\omega)$	differential operators
$(f * g)(t)$	$F(\omega)G(\omega)$	filtering and LTI systems
$r_{xy}(\tau)$	$X(\omega)\overline{Y(\omega)}$	detection and delay estimation

The useful property is the one that converts the difficult operation into the simplest frequency-domain calculation.

Choosing a representation

- **Periodic signal:** Fourier series gives discrete harmonic coefficients.
- **Nonperiodic signal with suitable decay or finite energy:** Fourier transform gives a continuous spectrum.
- **Causal initial-value problem or exponential growth:** Laplace transforms often provide a suitable convergence region and handle initial data directly.

Connection to the next lecture

The bilateral Laplace transform uses $s = \sigma + \mathrm{i}\omega$. Its value at $s = \mathrm{i}\omega$ agrees with the Fourier transform when the defining integral converges there.

A practical transform workflow

1. Identify periodicity, causality, symmetry, and growth.
2. Choose Fourier series, Fourier transform, or Laplace transform.
3. State the transform convention and convergence assumptions.
4. Exploit symmetry and transform properties before integrating.
5. Solve algebraically in the transform domain.
6. Invert and check dimensions, limits, and qualitative behavior.

Summary

- Integral transforms represent a function through a kernel and require an inverse synthesis rule.
- Fourier series resolve periodic signals into discrete harmonics.
- Half-range sine and cosine constructions encode boundary symmetry.
- Fourier transforms produce continuous spectra for nonperiodic signals.
- Shifts, scaling, differentiation, and convolution have simple frequency-domain forms.
- Energy and filtering interpretations make Fourier analysis central to engineering.

Looking ahead to Laplace transforms

Fourier analysis uses the oscillatory kernel $\mathrm{e}^{-\mathrm{i}\omega t}$. Laplace analysis adds an exponential weight:

$$\mathrm{e}^{-st} = \mathrm{e}^{-\sigma t}\mathrm{e}^{-\mathrm{i}\omega t}, \qquad s = \sigma + \mathrm{i}\omega.$$

What carries forward

Linearity, transform pairs, differentiation, and convolution.

What must be added

A region of convergence and explicit treatment of initial conditions.

L10 develops these tools for causal differential equations and transient responses.

Next $\gg$ Lecture 10 — Foundations of Laplace Transform