L10 — Foundations of Laplace Transform

Engineering Mathematics I

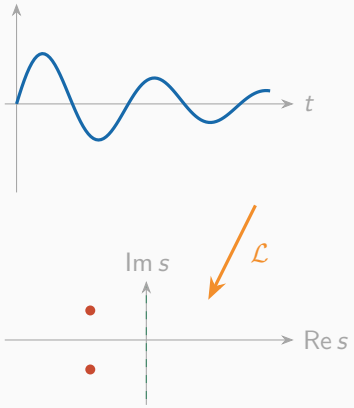
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Road map

- Definition of the Laplace transform
- Convergence and region of convergence
- Basic transform pairs and interpretation
- Poles and Time Behavior
- An Initial-Value Problem
- Bilateral Transforms: An Extension
- A Unified View



Definition of the Laplace transform

A signal beyond the ordinary Fourier integral

In L09, Fourier analysis described suitable signals through their frequencies. Consider instead $f(t) = \mathrm{e}^{t}$ for $t \geq 0$, extended by zero for $t < 0$.

$$\int_{0}^{\infty} \mathrm{e}^{t}\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t$$

does not converge as an ordinary improper integral.

An exponential weight

Multiplying by e^{-2t} gives $\mathrm{e}^{t}\mathrm{e}^{-2t} = \mathrm{e}^{-t}$, which is integrable.

We will vary the strength of this weight and keep it as part of the transform.

From Fourier frequency to complex frequency

L09 used the oscillatory kernel $\mathrm{e}^{-\mathrm{i}\omega t}$. Laplace analysis adds a real exponential weight:

$$\mathrm{e}^{-st} = \mathrm{e}^{-\sigma t}\mathrm{e}^{-\mathrm{i}\omega t}, \qquad s = \sigma + \mathrm{i}\omega.$$

- ω measures oscillation;
- σ controls exponential weighting;
- the extra degree of freedom admits growing signals and causal transients.

For signals considered only at $t \geq 0$, the Fourier comparison uses their zero extension to $t < 0$.

The unilateral Laplace transform

For a function given on $t \geq 0$, define

$$\boxed{F(s) = \mathcal{L}\{f\}(s) = \int_0^\infty f(t) \mathrm{e}^{-st}\, \mathrm{d}t}, \qquad s = \sigma + \mathrm{i}\omega.$$

The integral defines $F(s)$ wherever it converges. In this lecture, the *region of convergence* (ROC) refers to absolute convergence.

What the variables mean here

t is time. The real part σ controls weighting, while ω selects oscillation frequency.

For ordinary functions we use 0 as the lower limit and $f(0^+)$ for an initial value. Impulses at the origin require a separate convention, introduced with generalized inputs.

Worked example: the exponential and its domain

Let $f(t) = \mathrm{e}^{at}$ for $t \geq 0$, with real a.

Evaluate the improper integral

For $s \neq a$,

$$F(s) = \lim_{R\to\infty} \int_0^R \mathrm{e}^{-(s-a)t}\,\mathrm{d}t = \lim_{R\to\infty} \frac{1 - \mathrm{e}^{-(s-a)R}}{s-a}.$$

Since $|\mathrm{e}^{-(s-a)R}| = \mathrm{e}^{-(\sigma-a)R}$,

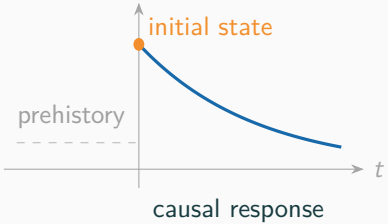
$$\boxed{\mathcal{L}\{\mathrm{e}^{at}\} = \frac{1}{s-a}, \qquad \operatorname{Re} s > a.}$$

If $\sigma = a$, the boundary term does not settle to a limit. If $\sigma < a$, it grows. The domain is part of the answer.

Why the unilateral transform fits engineering IVPs

Engineering signals are often switched on at $t = 0$ and observed for $t \geq 0$.

$$\mathcal{L}\{f\}(s) = \int_{0^-}^{\infty} f(t)\mathrm{e}^{-st}\,\mathrm{d}t.$$



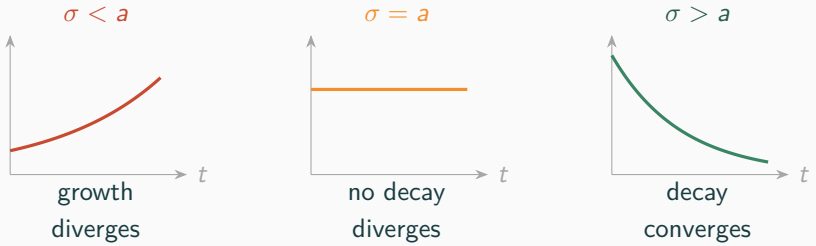
- prehistory enters through initial data;
- steps and impulses are handled naturally;
- causal systems lead to transfer functions.

For ordinary functions the lower limit may be written 0; 0^- includes an impulse located at the origin.

The real part of s controls convergence

For $f(t) = e^{at}$, the magnitude of the weighted integrand is

$$|f(t)e^{-st}| = e^{-(\sigma - a)t}.$$



The role of oscillation

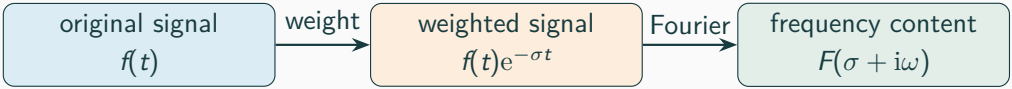
For this exponential, oscillation alone does not make the boundary integral converge. Moving right in the s-plane strengthens the damping.

The kernel separates growth and oscillation

With $s = \sigma + \mathrm{i}\omega$,

$$F(\sigma + \mathrm{i}\omega) = \int_0^\infty \left[f(t)\mathrm{e}^{-\sigma t}\right]\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t.$$

After zero extension to $t < 0$, this is the Fourier transform of the weighted signal, wherever the integral converges.



Convergence and Region of Convergence

Key points

- Interpret convergence as exponential damping.
- Check finite singularities separately from growth at infinity.
- Distinguish the transform integral from its continued algebraic formula.
- Recognize removable singularities for finite-duration signals.

The region of convergence

For the unilateral transform, define

$$\mathrm{ROC}(F) = \left\{ s : \int_0^\infty |f(t)| \mathrm{e}^{-(\mathrm{Re}\, s)t}\, \mathrm{d}t < \infty \right\}.$$

- Absolute convergence depends on $\mathrm{Re}\, s$, so the ROC is vertical in the s-plane.
- If convergence holds at a real part σ_1, it also holds at every $\sigma > \sigma_1$.
- The interior is a right half-plane (possibly the whole plane or empty). Boundary convergence needs a separate check.

A transform answer records both $F(s)$ and its ROC.

Worked example: a growing exponential

For $f(t) = \mathrm{e}^{at}$, $t \geq 0$,

$$F(s) = \int_0^\infty \mathrm{e}^{-(s-a)t}\,\mathrm{d}t = \boxed{\frac{1}{s-a}},$$

provided

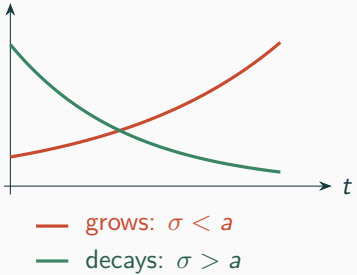
$$\mathrm{Re}(s-a) > 0 \quad \Longleftrightarrow \quad \mathrm{Re}\, s > a.$$

The pole $s = a$ lies on the boundary; the causal ROC is the half-plane to its right.

For $f(t) = \mathrm{e}^{at}$, the weighting $\mathrm{e}^{-\sigma t}$ produces

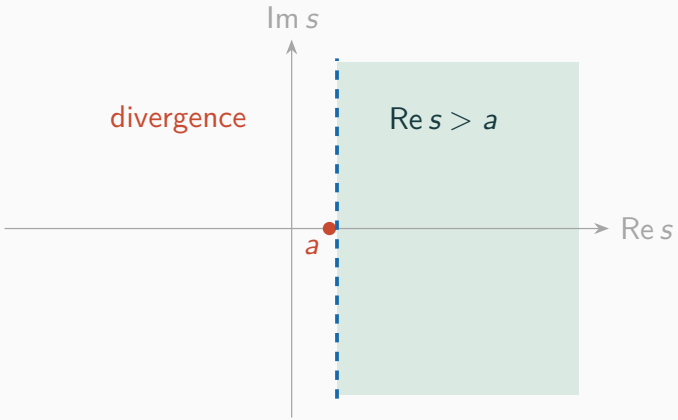
$$f(t)\mathrm{e}^{-\sigma t} = \mathrm{e}^{-(\sigma-a)t}.$$

Decay occurs only when $\sigma > a$.



The ROC is visible in the s-plane

For $f(t) = \mathrm{e}^{at}$ with $a > 0$, $F(s) = 1/(s - a)$.



A standard existence theorem

If f is piecewise continuous on every finite interval $[0, A]$ and is of exponential order α, meaning

$$|f(t)| \leq Me^{\alpha t} \qquad (t \geq T),$$

then $\mathcal{L}\{f\}(s)$ exists for

$$\operatorname{Re} s > \alpha.$$

These are sufficient conditions, not necessary conditions.

Why exponential order is sufficient

For $\sigma = \operatorname{Re} s > \alpha$,

$$|f(t)\mathrm{e}^{-st}| \le M\mathrm{e}^{-(\sigma-\alpha)t}.$$

Since

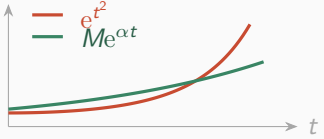
$$\int_T^\infty M\mathrm{e}^{-(\sigma-\alpha)t}\,\mathrm{d}t < \infty,$$

comparison proves absolute convergence of the tail. Piecewise continuity handles the integral on $[0, T]$.

Two separate checks: near finite points and at infinity

Growth is too fast: $f(t) = e^{t^2}$. For every fixed σ,

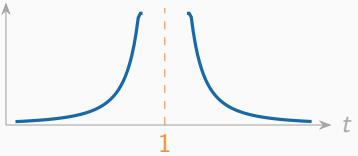
$$f(t)e^{-\sigma t} = e^{t^2 - \sigma t} \to \infty.$$



A finite singularity is not integrable:

$$f(t) = \frac{1}{(t-1)^2}.$$

Exponential weighting cannot repair the divergence at $t = 1$.



Existence requires both acceptable local behavior and controlled growth at infinity.

Worked example: a shifted unit box

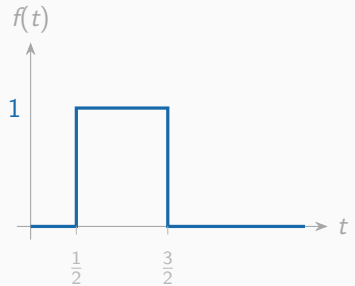
Let $f(t) = 1$ for $\frac{1}{2} < t < \frac{3}{2}$ and zero otherwise. Directly,

$$F(s) = \int_{1/2}^{3/2} \mathrm{e}^{-st}\,\mathrm{d}t = \boxed{\frac{\mathrm{e}^{-s/2} - \mathrm{e}^{-3s/2}}{s}}.$$

The apparent singularity is removable:

$$\lim_{s \to 0} F(s) = 1.$$

Hence F is entire: an integrable signal with finite time support converges for every $s \in$.



compact support

$\Rightarrow$ entire transform

Analyticity inside the ROC

Inside the absolute-convergence half-plane, exponential domination permits differentiation under the integral:

$$F^{(n)}(s) = (-1)^n \int_0^\infty t^n f(t) \mathrm{e}^{-st} \, \mathrm{d}t.$$

Thus F is analytic in the interior of the ROC.

Integral versus continued formula

For $f(t) = 1$, the integral converges only when $\operatorname{Re} s > 0$. The formula $1/s$ is also defined at $s = -1$, but its value there is not the value of the defining integral.

A pole cannot lie inside the ROC. Being away from every pole is not sufficient for convergence.

Convergence checklist

Before accepting a transform:

1. identify discontinuities and local integrability;
2. estimate growth as $t \to \infty$;
3. compare $\operatorname{Re} s$ with the exponential growth rate;
4. locate poles and removable singularities;
5. state the resulting ROC explicitly.

A transform table gives the formula; analysis supplies the domain.

Elementary Transform Pairs

Key points

- Build powers using integration by parts.
- Build sine, cosine, and hyperbolic functions from exponentials.
- Combine known pairs before integrating again.
- Check the ROC and dimensions with each answer.

Linearity and convergence

Whenever both integrals converge absolutely,

$$\mathcal{L}\{Af + Bg\} = A\mathcal{L}\{f\} + B\mathcal{L}\{g\}.$$

The intersection of the component ROCs is a guaranteed convergence region for the sum.

Cancellation can enlarge the ROC

If $f(t) = \mathrm{e}^{t}$ and $g(t) = -\mathrm{e}^{t}$, each component requires $\operatorname{Re} s > 1$. Their sum is zero, whose transform converges everywhere.

Simplify the signal or transform before claiming that an intersection is the exact ROC.

The unit step and decay

For $t \geq 0$ and $a > 0$, the exponential calculation gives

$$1 \longleftrightarrow \frac{1}{s}, \qquad \operatorname{Re} s > 0,$$

$$\mathrm{e}^{-at} \longleftrightarrow \frac{1}{s+a}, \qquad \operatorname{Re} s > -a.$$

The unilateral constant 1 is the causal unit step after zero extension. Faster decay moves the boundary farther left.

Powers of time: a recurrence from integration by parts

For an integer $n \geq 1$ and $\operatorname{Re} s > 0$, integration by parts gives

$$\begin{aligned} I_n(s) &= \int_0^\infty t^n \mathrm{e}^{-st}\,\mathrm{d}t \\ &= \left[-\frac{t^n \mathrm{e}^{-st}}{s}\right]_0^\infty + \frac{n}{s}\int_0^\infty t^{n-1}\mathrm{e}^{-st}\,\mathrm{d}t = \frac{n}{s} I_{n-1}(s). \end{aligned}$$

Starting with $I_0(s) = 1/s$, we obtain

$$\boxed{\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad n \geq 0, \quad \operatorname{Re} s > 0.}$$

First entries

$1 \longleftrightarrow 1/s$, $t \longleftrightarrow 1/s^2$, and $t^2 \longleftrightarrow 2/s^3$.

Sine and cosine from exponential pairs

Let $b > 0$. Euler's identities give

$$\cos bt = \frac{\mathrm{e}^{\mathrm{i}bt} + \mathrm{e}^{-\mathrm{i}bt}}{2}, \qquad \sin bt = \frac{\mathrm{e}^{\mathrm{i}bt} - \mathrm{e}^{-\mathrm{i}bt}}{2\mathrm{i}}.$$

For $\operatorname{Re} s > 0$, linearity therefore gives

$$\mathcal{L}\{\cos bt\} = \frac{1}{2}\left(\frac{1}{s - \mathrm{i}b} + \frac{1}{s + \mathrm{i}b}\right) = \boxed{\frac{s}{s^2 + b^2}},$$

$$\mathcal{L}\{\sin bt\} = \frac{1}{2\mathrm{i}}\left(\frac{1}{s - \mathrm{i}b} - \frac{1}{s + \mathrm{i}b}\right) = \boxed{\frac{b}{s^2 + b^2}}.$$

For complex s, these formulas follow from sums and differences, not from taking real and imaginary parts.

Hyperbolic functions

For real $a > 0$,

$$\cosh at = \frac{\mathrm{e}^{at} + \mathrm{e}^{-at}}{2}, \qquad \sinh at = \frac{\mathrm{e}^{at} - \mathrm{e}^{-at}}{2},$$

linearity gives

$$\mathcal{L}\{\cosh at\} = \frac{s}{s^2 - a^2}, \qquad \mathcal{L}\{\sinh at\} = \frac{a}{s^2 - a^2},$$

Both exponentials must converge, giving the exact ROC $\operatorname{Re} s > a$.

Worked example: linearity builds a transform

If $f \longleftrightarrow F$ and $g \longleftrightarrow G$, then

$$af + bg \longleftrightarrow aF + bG.$$

Example:

$$f(t) = 3t^2 - 2\mathrm{e}^{-4t} + 5\sin(2t)$$

gives

$$F(s) = \boxed{\frac{6}{s^3} - \frac{2}{s+4} + \frac{10}{s^2+4}}.$$

Here the exact ROC is $\operatorname{Re} s > 0$. The polynomial and oscillatory components do not cancel, so the intersection is not enlarged.

Worked example: an exponential weight shifts s

For $f(t) = te^{-at}$ with $a > 0$,

$$F(s) = \int_0^\infty te^{-(s+a)t}\,\mathrm{d}t.$$

Using the power pair with s replaced by $s + a$,

$$\boxed{F(s) = \frac{1}{(s+a)^2}}, \qquad \operatorname{Re} s > -a.$$

This previews the frequency-shifting property developed in L11.

Worked example: a damped sinusoid

For $a > 0$ and $\omega > 0$,

$$f(t) = \mathrm{e}^{-at}\sin(\omega t)$$

has transform

$$\boxed{F(s) = \frac{\omega}{(s+a)^2 + \omega^2}}, \qquad \operatorname{Re} s > -a.$$

The poles move from $\pm \mathrm{i}\omega$ to $-a \pm \mathrm{i}\omega$: damping shifts the oscillatory modes into the left half-plane.

Units and dimensional consistency

Because the exponent st must be dimensionless,

$$[s] = [\text{time}]^{-1}.$$

If f has units U, then

$$[F] = U \cdot [\text{time}].$$

Dimensional checks catch common missing factors in scaling, shifting, and transform-table calculations.

A compact starter table

Here $n \geq 0$ is an integer, a is real, and $\omega > 0$.

$f(t)$	$F(s)$	unilateral ROC
1	$1/s$	$\operatorname{Re} s > 0$
t^n	$n!/s^{n+1}$	$\operatorname{Re} s > 0$
e^{at}	$1/(s-a)$	$\operatorname{Re} s > a$
$\cos \omega t$	$s/(s^2+\omega^2)$	$\operatorname{Re} s > 0$
$\sin \omega t$	$\omega/(s^2+\omega^2)$	$\operatorname{Re} s > 0$

How to use a transform table?

1. rewrite the function in standard components;
2. factor constants and expose shifts;
3. identify the closest known transform pair;
4. apply properties before integrating from scratch;
5. state the ROC and verify dimensions.

Direct integration remains the definition and the final check, but it is rarely the fastest routine method.

Extension: fractional powers and the gamma function

For real $\nu > -1$, define

$$\Gamma(\nu+1) = \int_0^\infty u^\nu \mathrm{e}^{-u}\,\mathrm{d}u.$$

For real $s > 0$, the substitution $u = st$ gives the power transform. Analytic continuation within $\operatorname{Re} s > 0$ then gives

$$\boxed{\mathcal{L}\{t^\nu\} = \frac{\Gamma(\nu+1)}{s^{\nu+1}}, \qquad \operatorname{Re} s > 0.}$$

Here $s^{\nu+1} = \exp((\nu+1)\operatorname{Log} s)$ uses $-\pi/2 < \arg s < \pi/2$.

A sufficient condition need not be necessary

$\mathcal{L}\{t^{-1/2}\} = \sqrt{\pi}/\sqrt{s}$ exists although $t^{-1/2}$ is unbounded at zero. Its local singularity is integrable.

Poles and Time Behavior

Simple and repeated poles describe different modes

For right-sided rational transform pairs,

$$\frac{1}{s-p} \longleftrightarrow \mathrm{e}^{pt}, \qquad \frac{1}{(s-p)^m} \longleftrightarrow \frac{t^{m-1}}{(m-1)!}\mathrm{e}^{pt}.$$

Write a pole as $p = \alpha + \mathrm{i}\beta$:

$$\mathrm{e}^{pt} = \mathrm{e}^{\alpha t}(\cos\beta t + \mathrm{i}\sin\beta t).$$

The real part controls the exponential envelope; the imaginary part controls oscillation.

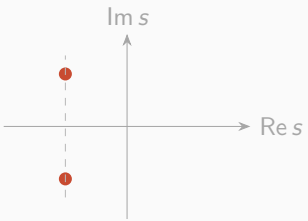
Multiplicity matters

A simple imaginary-axis pole gives a sustained mode. A repeated one adds a polynomial factor and can produce growth. These statements describe signal modes, not a complete system-stability test.

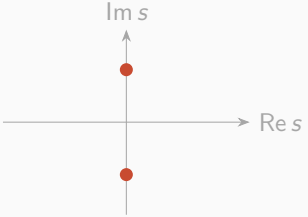
Pole location classifies natural modes

For a real rational transform, nonreal poles occur in conjugate pairs. The sketches here show simple poles.

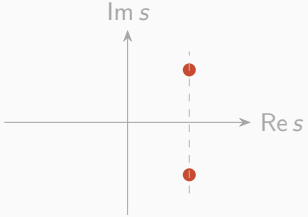
Decay: $\sigma < 0$



Sustained: $\sigma = 0$



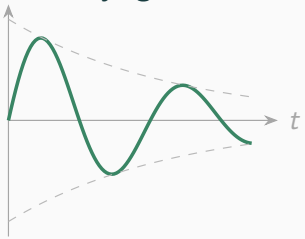
Growth: $\sigma > 0$



Interpretation: horizontal pole position determines decay, persistence, or growth.

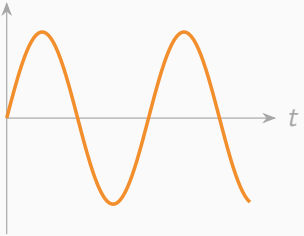
The same classification appears in time

Decaying oscillation



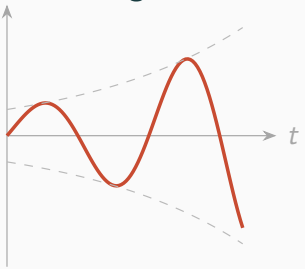
$e^{\sigma t} \sin \omega t, \quad \sigma < 0$

Sustained oscillation



$\sin \omega t, \quad \sigma = 0$

Growing oscillation



$e^{\sigma t} \sin \omega t, \quad \sigma > 0$

These simple-pole examples connect exponential envelopes with pole position.

When Laplace analysis recovers Fourier analysis

Let $f_+(t) = f(t)$ for $t \geq 0$ and 0 for $t < 0$. At a fixed σ inside the ROC,

$$F(\sigma + \mathrm{i}\omega) = \mathcal{F}\{f_+(t)\mathrm{e}^{-\sigma t}\}(\omega).$$

If the imaginary axis belongs to the absolute ROC, then

$$F(\mathrm{i}\omega) = \mathcal{F}\{f_+\}(\omega).$$

Two signals from this lecture

e^{-at} with $a > 0$ has ROC $\operatorname{Re} s > -a$, which includes that axis. For 1, the ROC is $\operatorname{Re} s > 0$, and the ordinary Fourier integral does not converge.

Outside the absolute ROC, this criterion gives no conclusion about conditional or generalized Fourier transforms.

An Initial-Value Problem

What the unilateral transform records

An initial-value problem specifies the state at $t = 0$ and asks for the solution at $t > 0$.

- The integral uses only future values, $t \geq 0$.
- Derivative formulas carry the initial data into the transformed equation.
- The solution need not be zero before 0; that prehistory is outside this calculation.

Causality and initial data

A zero extension is useful when comparing with a bilateral transform. It must not be confused with asserting that a nonzero initial state was created from zero without an input.

A derivative introduces the initial value

Let y be locally absolutely continuous, with y and y' of exponential order. For sufficiently large $\operatorname{Re} s$, integration by parts gives

$$\begin{aligned}\mathcal{L}\{y'\} &= \left[y(t)\mathrm{e}^{-st}\right]_{0^+}^{\infty} + s\int_0^{\infty} y(t)\mathrm{e}^{-st}\,\mathrm{d}t \\ &= \boxed{sY(s) - y(0^+)}.\end{aligned}$$

Why this helps an initial-value problem

The boundary term retains the initial value while the derivative becomes an algebraic expression.

This formula uses the ordinary derivative for $t > 0$. It does not include a distributional impulse from a zero extension at 0.

Worked example: a complete initial-value problem

Solve $y' + 2y = 1$ for $t > 0$, with $y(0^+) = 3$.

Transform, solve, and invert

$$(sY - 3) + 2Y = \frac{1}{s}, \qquad Y = \frac{3 + 1/s}{s+2} = \frac{1}{2s} + \frac{5}{2(s+2)}.$$

Using the constant and exponential pairs,

$$\boxed{y(t) = \frac{1}{2} + \frac{5}{2}\mathrm{e}^{-2t}, \qquad t \geq 0.}$$

Checks: $y(0^+) = 3$ and $y' + 2y = 1$. The exact ROC of Y is $\operatorname{Re} s > 0$. The constant input creates a constant steady response, while the transient decays.

Bilateral Transforms: An Extension

Extension: the bilateral Laplace transform

When a signal is defined on both sides of zero, use

$$F_{\mathrm{b}}(s) = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-st}\,\mathrm{d}t.$$

- Convergence must now hold as $t \to +\infty$ and $t \to -\infty$.
- A causal zero extension gives the same integral as the unilateral transform.
- A general two-sided signal can have a strip or a left half-plane as its ROC.

We now compare with the unilateral case to understand why an algebraic formula alone can be ambiguous bilaterally.

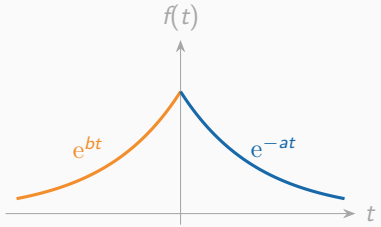
Two-sided exponentials produce strips

Let

$$f(t) = \begin{cases} \mathrm{e}^{-at}, & t \geq 0, \\ \mathrm{e}^{bt}, & t < 0, \end{cases} \qquad a, b > 0.$$

The positive-time integral requires $\operatorname{Re} s > -a$; the negative-time integral requires $\operatorname{Re} s < b$. Hence

$$\boxed{-a < \operatorname{Re} s < b}.$$



The two integrals give $F_{\mathrm{b}}(s) = 1/(s+a) + 1/(b-s)$ on this strip.

The ROC strip is an intersection of two half-planes

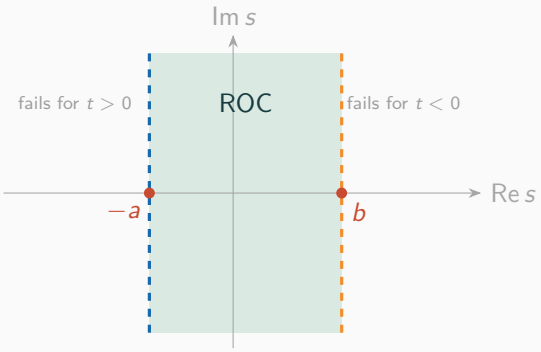
Positive time requires

$$\operatorname{Re} s > -a,$$

while negative time requires

$$\operatorname{Re} s < b.$$

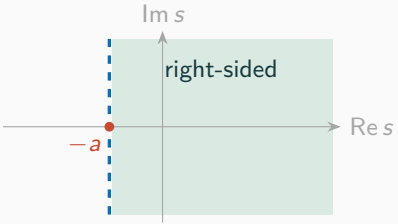
Their intersection is the vertical strip $-a < \operatorname{Re} s < b$.



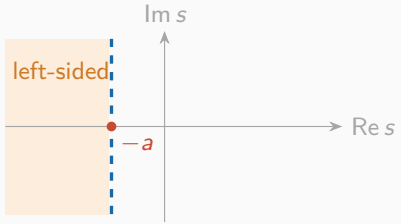
The algebraic formula alone is insufficient

Let $a > 0$ and let u be the unit step. Bilaterally, $F(s) = 1/(s+a)$ has two interpretations:

$\mathrm{e}^{-at}u(t), \qquad \operatorname{Re} s > -a.$



$-\mathrm{e}^{-at}u(-t), \qquad \operatorname{Re} s < -a.$



The algebraic expression is identical; the ROC supplies the missing time-direction information.

ROC rules for rational transforms

For a nonzero, strictly proper rational bilateral transform with a nonempty ROC, after all pole cancellations:

- the ROC contains no poles;
- the ROC is a connected vertical strip or half-plane;
- a right-sided signal has an ROC to the right of the rightmost pole;
- a left-sided signal has an ROC to the left of the leftmost pole;
- a two-sided signal has an ROC between poles.

For the unilateral rational pairs here, the ROC lies to the right of the rightmost remaining pole.

A Unified View

Practice

1. Find the transform and exact ROC of $2\mathrm{e}^{3t} + t$ for $t \geq 0$.
2. Does its zero extension have an absolutely convergent Fourier transform?
3. What time function corresponds to $1/(s+2)^2$ unilaterally?
4. For the shifted unit box, why is $s = 0$ not a pole?

Answer check

$F(s) = 2/(s-3) + 1/s^2$, ROC $\operatorname{Re} s > 3$. The imaginary axis is excluded. The inverse of $1/(s+2)^2$ is $t\mathrm{e}^{-2t}$. The box has $F(0) = 1$ by its integral, so the apparent singularity is removable.

Common interpretation errors

- A pole is not automatically unstable; causality and the chosen ROC matter.
- A canceled pole may be removable and must be simplified before interpreting modes.
- The transform variable s is not time.
- The ROC is not obtained by merely deleting poles from the plane.
- A transform formula must use the same convention as the inverse or table.

A complete forward-transform workflow

1. identify causality, discontinuities, and growth;
2. choose unilateral or bilateral definition;
3. establish a convergence region;
4. simplify the signal into standard components;
5. compute the transform analytically;
6. locate poles, zeros, and removable singularities;
7. check dimensions and limiting behavior.

Summary

1. Exponential weighting extends the frequency-domain method to growing signals.
2. The unilateral transform integrates over $t \geq 0$; its derivative formula includes the initial value.
3. Local integrability and tail behavior determine convergence. Record the ROC with the formula.
4. Elementary pairs follow from the exponential integral, linearity, and integration by parts.
5. Simple poles describe exponential modes; repeated poles add polynomial factors.
6. Bilateral transforms require attention to both time directions and their shared ROC.

Next $\gg$ Lecture 11 — Laplace Transform Properties