L11 — Laplace Transform Properties

Engineering Mathematics I

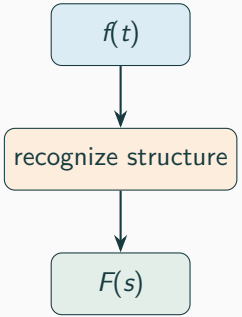
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Road map

- Linearity and scaling
- Shifts
- Differentiation and integration
- Piecewise and periodic inputs
- Convolution theorem



Properties replace repeated integration

L10 defined

$$F(s) = \int_0^\infty f(t)\mathrm{e}^{-st}\,\mathrm{d}t.$$

Operational properties let us transform a modified signal by transforming a simpler parent signal first.

Strategy

Recognize structure before integrating: combination, scale, shift, derivative, integral, periodicity, or convolution.

Linearity and Scaling

Linearity

If $f(t) \longleftrightarrow F(s)$ and $g(t) \longleftrightarrow G(s)$, then

$$\boxed{\mathcal{L}\{af(t) + bg(t)\} = aF(s) + bG(s)}.$$

It follows directly from linearity of integration. The ROC contains at least the intersection of the component ROCs; pole cancellation may enlarge it after simplification.

Example

For

$$f(t) = 4 - 3t + 2\cos(5t),$$

linearity and the starter table give

$$F(s) = \frac{4}{s} - \frac{3}{s^2} + \frac{2s}{s^2 + 25}, \qquad \operatorname{Re} s > 0.$$

No new improper integral is required.

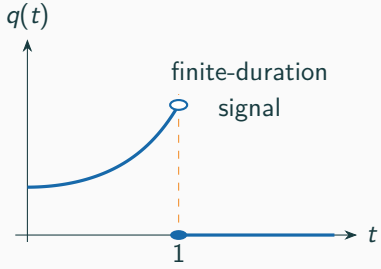
A convergence trap in linearity

Consider

$$q(t) = \mathrm{e}^{t^2} - u(t-1)\mathrm{e}^{t^2} = \mathrm{e}^{t^2}\big[1 - u(t-1)\big].$$

Each displayed term grows too rapidly to possess a Laplace transform, but their *combination* has finite support:

$$q(t) = 0 \qquad (t > 1).$$



Lesson

Simplify the time-domain signal before applying linearity term by term.

Time scaling

For $a > 0$,

$$\boxed{\mathcal{L}\{f(at)\}(s) = \frac{1}{a} F\left(\frac{s}{a}\right)}.$$

Derivation follows from $u = at$:

$$\int_0^\infty f(at) \mathrm{e}^{-st}\, \mathrm{d}t = \frac{1}{a} \int_0^\infty f(u) \mathrm{e}^{-(s/a)u}\, \mathrm{d}u.$$

Time compression expands the characteristic scale in the s-domain.

Example

Since

$$\sin t \longleftrightarrow \frac{1}{s^2 + 1},$$

time scaling gives

$$\sin(at) \longleftrightarrow \frac{1}{a} \frac{1}{(s/a)^2 + 1} = \boxed{\frac{a}{s^2 + a^2}}.$$

The pole pair moves from $\pm \mathrm{i}$ to $\pm \mathrm{i}a$.

Amplitude scaling is not time scaling

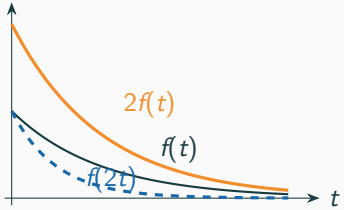
Compare

$$af(t) \longleftrightarrow aF(s)$$

with

$$f(at) \longleftrightarrow \frac{1}{a}F(s/a).$$

Amplitude scaling changes height only; time scaling changes duration, transform amplitude, pole locations, and frequency content.



Property and ROC must travel together

If

$$F(s) \text{ converges for } \boxed{\operatorname{Re} s > \alpha}$$

then

$$\frac{1}{a} F(s/a) \text{ converges for } \boxed{\operatorname{Re}(s/a) > \alpha}$$

i.e.,

$$\boxed{\operatorname{Re} s > a\alpha} \text{ when } a > 0.$$

Check

Transform properties act on both the formula and its convergence region.

Shifts

Exponential shift in the transform variable

If $f(t) \longleftrightarrow F(s)$, then

$$\boxed{\mathrm{e}^{at} f(t) \longleftrightarrow F(s-a)}.$$

Indeed,

$$\int_0^\infty \mathrm{e}^{at} f(t) \mathrm{e}^{-st}\,\mathrm{d}t = \int_0^\infty f(t) \mathrm{e}^{-(s-a)t}\,\mathrm{d}t = F(s-a).$$

Multiplication by an exponential translates poles and the ROC horizontally.

Worked example: damped oscillation

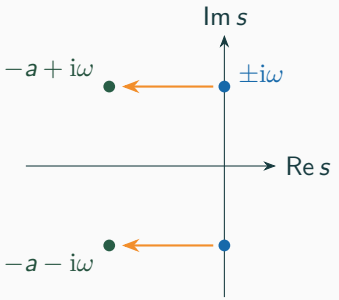
From

$$\sin \omega t \longleftrightarrow \frac{\omega}{s^2 + \omega^2},$$

substitute $s \mapsto s + a$:

$$\boxed{\mathrm{e}^{-at} \sin \omega t \longleftrightarrow \frac{\omega}{(s+a)^2 + \omega^2}}.$$

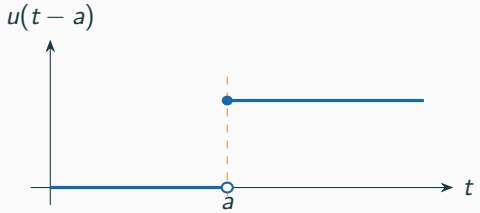
The poles shift left by a, representing exponential damping.



The unit step describes switching

$$u(t-a) = \begin{cases} 0, & t < a, \\ 1, & t > a. \end{cases} \qquad \boxed{u(t-a) \longleftrightarrow \frac{\mathrm{e}^{-as}}{s}}.$$

Its value at $t = a$ does not affect the Laplace integral.



Time delay

For $a > 0$,

$$\boxed{u(t-a)f(t-a) \longleftrightarrow \mathrm{e}^{-as}F(s)}.$$

The unit step enforces causality at the new start time. A delay multiplies the transform by an exponential and does not move the original poles of F.

Deriving the time-delay property

$$\mathcal{L}\{u(t-a)f(t-a)\} = \int_0^\infty u(t-a)f(t-a)\mathrm{e}^{-st}\,\mathrm{d}t = \int_a^\infty f(t-a)\mathrm{e}^{-st}\,\mathrm{d}t.$$

Set $\tau = t - a$:

$$\mathcal{L}\{u(t-a)f(t-a)\} = \mathrm{e}^{-as}\int_0^\infty f(\tau)\mathrm{e}^{-s\tau}\,\mathrm{d}\tau = \mathrm{e}^{-as}F(s).$$

The step and shifted argument must appear together.

Worked example: a delayed ramp

Since $t \longleftrightarrow 1/s^2$,

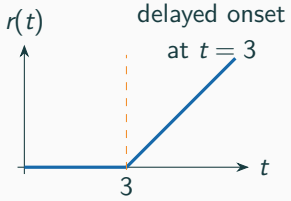
$$(t-3)u(t-3) \longleftrightarrow \boxed{\frac{\mathrm{e}^{-3s}}{s^2}}.$$

In contrast,

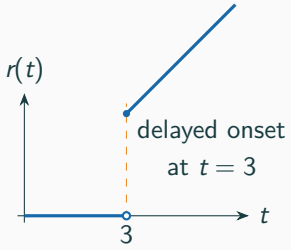
$$t\,u(t-3) = \big[(t-3)+3\big]\,u(t-3)$$

transforms to

$$\mathrm{e}^{-3s}\left(\frac{1}{s^2}+\frac{3}{s}\right).$$



$(t-3)u(t-3)$



$(t-3)u(t-3)$

Representing a switched signal

Suppose

$$f(t) = \begin{cases} t, & 0 \le t < 2, \\ 4 - t, & t \ge 2. \end{cases}$$

Write it as

$$f(t) = t + u(t-2)\big[(4-t) - t\big] = t - 2(t-2)u(t-2).$$

Therefore

$$F(s) = \boxed{\frac{1}{s^2} - \frac{2\mathrm{e}^{-2s}}{s^2}}.$$

Delay versus exponential weighting

Time operation	Transform operation	Main effect
$\mathrm{e}^{at}f(t)$	$F(s-a)$	shifts poles and ROC
$u(t-a)f(t-a)$	$\mathrm{e}^{-as}F(s)$	delays signal onset

These two shift theorems are commonly confused because both involve exponentials.

Differentiation and Integration

Time differentiation introduces initial values

First-order derivative

Integration by parts gives

$$\boxed{\mathcal{L}\{f'(t)\} = sF(s) - f(0^+)}.$$

The boundary term at infinity vanishes under the convergence assumptions, while the lower boundary carries the initial value.

Higher-order derivatives

Repeated application yields

$$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0^+) - \cdots - f^{(n-1)}(0^+).$$

This is the central reason the unilateral transform is effective for initial-value problems.

Worked example: verify with a known function

Let

$$f(t) = \sin \omega t.$$

Hence, $f(0) = 0$ and

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0) = s\frac{\omega}{s^2 + \omega^2} - 0 = \boxed{\frac{\omega s}{s^2 + \omega^2}}$$

Moreover,

$$f'(t) = \omega \cos \omega t.$$

Then

$$\mathcal{L}\{f'(t)\} = \mathcal{L}\{\omega \cos \omega t\} = \boxed{\frac{\omega s}{s^2 + \omega^2}}.$$

Integration in time

Let

$$g(t) = \int_0^t f(\tau)\,\mathrm{d}\tau.$$

Since $g'(t) = f(t)$ and $g(0) = 0$,

$$\boxed{\mathcal{L}\{g(t)\} = \frac{F(s)}{s}}.$$

Interpretation

Time integration introduces an additional pole at the origin.

Multiplication by t differentiates F

Differentiating the defining integral with respect to s gives

$$F'(s) = -\int_0^\infty t f(t) \mathrm{e}^{-st}\,\mathrm{d}t.$$

Hence

$$\boxed{\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s)}.$$

Multiplication by time becomes differentiation in the transform domain.

Example

From

$$\sin \omega t \longleftrightarrow F(s) = \frac{\omega}{s^2 + \omega^2},$$

$$\mathcal{L}\{t \sin \omega t\} = -F'(s) = \boxed{\frac{2\omega s}{(s^2 + \omega^2)^2}}.$$

Differentiating a rational transform is simpler than evaluating a new improper integral.

Worked example: a second (s)-derivative

Starting from

$$\cos \omega t \longleftrightarrow F(s) = \frac{s}{s^2 + \omega^2},$$

the $n = 2$ case gives

$$\mathcal{L}\{t^2 \cos \omega t\} = F''(s).$$

Differentiating twice and simplifying,

$$\boxed{\mathcal{L}\{t^2 \cos \omega t\} = \frac{2s(s^2 - 3\omega^2)}{(s^2 + \omega^2)^3}}.$$

The theorem converts multiplication by a time polynomial into an algebraic differentiation workflow.

Division by t integrates F

Under appropriate integrability conditions,

$$\boxed{\mathcal{L}\left\{\frac{f(t)}{t}\right\}(s) = \int_s^\infty F(\sigma)\,\mathrm{d}\sigma}.$$

This follows by reversing the order of integration. The behavior of $f(t)/t$ near $t = 0$ must be checked before applying the rule.

Property-selection guide

Visible structure in $f(t)$	First property to try
sum of known functions	linearity
$f(at)$	time scaling
$\mathrm{e}^{at} f(t)$	s-shift
delayed onset	unit-step delay
$f'(t)$ with initial data	derivative rule
$\int_0^t f(\tau)\,\mathrm{d}\tau$	time integration
$t^n f(t)$	differentiation with respect to s
$f(t)/t^n$	integration with respect to s

Piecewise and Periodic Inputs

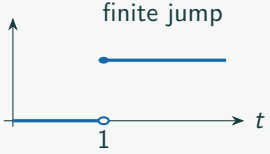
Piecewise continuity

A finite number of jumps on every bounded interval does not prevent a Laplace transform. Split the defining integral at the discontinuities:

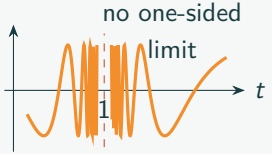
$$F(s) = \int_0^{a_1} + \int_{a_1}^{a_2} + \cdots + \int_{a_n}^{\infty} f(t)\mathrm{e}^{-st}\,\mathrm{d}t.$$

Unit-step notation usually produces a more reusable result.

A jump is allowed; infinite local oscillation is not



Piecewise continuous



Not piecewise continuous near ($t = 1$)

Laplace-transform existence allows isolated finite jumps, but not an essential discontinuity with no finite one-sided limit.

Pulses are differences of steps

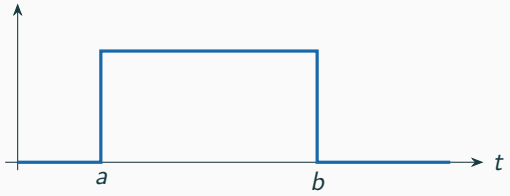
A finite pulse on $[a, b]$ can be written

$$u(t-a) - u(t-b),$$

so

$$\mathcal{L}\{u(t-a) - u(t-b)\} = \boxed{\frac{\mathrm{e}^{-as} - \mathrm{e}^{-bs}}{s}}.$$

Piecewise-linear pulses follow by combining delayed ramps.



The Dirac impulse

The impulse is characterized by

$$\int_{-\infty}^{\infty} \delta(t-a)\phi(t)\,\mathrm{d}t = \phi(a).$$

For $a \geq 0$,

$$\boxed{\mathcal{L}\{\delta(t-a)\} = \mathrm{e}^{-as}}.$$

It models an input with finite total effect concentrated at one instant; full distribution theory is outside the course.

Example

For

$$f(t) = 3u(t-2) + 5\delta(t-4),$$

$$F(s) = \boxed{\frac{3\mathrm{e}^{-2s}}{s} + 5\mathrm{e}^{-4s}}.$$

The step contributes a sustained level after $t = 2$; the impulse contributes an instantaneous event at $t = 4$.

Laplace transform of a periodic function

If $f(t+T) = f(t)$, split $[0, \infty)$ into periods:

$$F(s) = \sum_{k=0}^{\infty} \mathrm{e}^{-ksT} \int_0^T f(t)\mathrm{e}^{-st}\,\mathrm{d}t.$$

Summing the geometric series gives

$$\boxed{F(s) = \frac{\int_0^T f(t)\mathrm{e}^{-st}\,\mathrm{d}t}{1-\mathrm{e}^{-sT}}}, \qquad \operatorname{Re} s > 0$$

for bounded piecewise-continuous periodic signals.

Why this works

Each later period contributes the same one-period integral multiplied by e^{-sT} again, so the contributions form a geometric series.

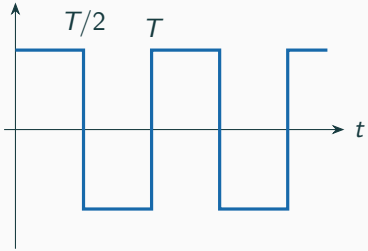
Worked example: periodic square wave

Let $f(t) = 1$ on $0 < t < T/2$ and $f(t) = -1$ on $T/2 < t < T$, repeated periodically. Then

$$\int_0^T f(t)\mathrm{e}^{-st}\,\mathrm{d}t = \frac{(1-\mathrm{e}^{-sT/2})^2}{s}.$$

Hence

$$F(s) = \boxed{\frac{1-\mathrm{e}^{-sT/2}}{s(1+\mathrm{e}^{-sT/2})}} = \frac{1}{s}\tanh\left(\frac{sT}{4}\right).$$



Periodic repetition produces a pole pattern

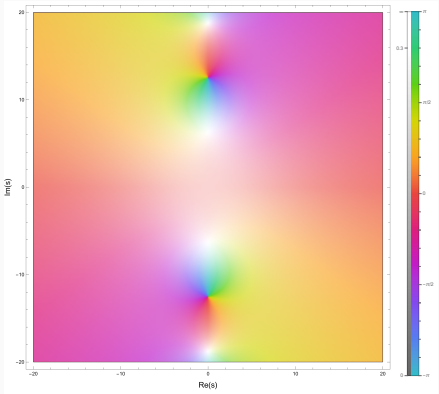
For the unit-period square wave,

$$F(s) = \frac{1}{s}\tanh\left(\frac{s}{4}\right).$$

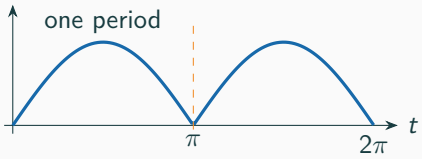
The periodic denominator creates the repeated imaginary-axis pattern. Focus on recurring singular features; the colors encode phase.

Observe

Periodicity in time leaves a regularly spaced signature in complex frequency.



Worked example: full-wave rectification changes the period



The signal

$$f(t) = |\sin t|$$

repeats after $T = \pi$, not 2π. On one period, $f(t) = \sin t$ for $0 \le t \le \pi$.

Engineering interpretation

Full-wave rectification flips every negative half-cycle upward, doubling the repetition frequency.

The one-period integral is

$$\int_0^{\pi} \sin t\, \mathrm{e}^{-st}\, \mathrm{d}t = \frac{1 + \mathrm{e}^{-\pi s}}{s^2 + 1}.$$

Dividing by $1 - \mathrm{e}^{-\pi s}$ gives

$$\boxed{\mathcal{L}\{|\sin t|\} = \frac{1}{s^2 + 1} \coth\!\left(\frac{\pi s}{2}\right)}, \qquad \operatorname{Re} s > 0.$$

Two quick checks

The period is π, and as $s \to \infty$ the transform behaves like $1/s^2$, consistent with $|\sin t| \sim t$ near the origin.

Periodic-signal checklist

1. Find the smallest period T.
2. Write the signal only for $0 \leq t < T$.
3. Evaluate the one-period integral.
4. Divide by $1 - \mathrm{e}^{-sT}$.
5. Check the result against the behavior near $t = 0$ when possible.

Reminder

Do not integrate many repeated periods separately.

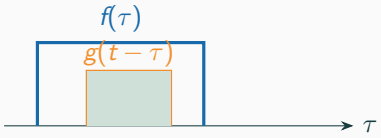
Convolution Theorem

Unilateral convolution

For causal functions,

$$(f * g)(t) = \int_0^t f(\tau) g(t-\tau)\,\mathrm{d}\tau.$$

At each time t, slide the reversed-and-shifted copy $g(t-\tau)$ across $f(\tau)$ and add the overlapping product.



Convolution combines all delayed overlaps of f and g. It is commutative and associative under standard integrability conditions.

Convolution theorem

$$\boxed{\mathcal{L}\{f(t) * g(t)\} = F(s)G(s)}.$$

Conversely,

$$\mathcal{L}^{-1}\{F(s)G(s)\} = f(t) * g(t).$$

Multiplication in the transform domain replaces an integral operation in time.

Deriving the convolution theorem

Start with

$$\int_0^\infty \mathrm{e}^{-st} \int_0^t f(\tau) g(t-\tau)\,\mathrm{d}\tau\,\mathrm{d}t.$$

Set $u = t - \tau$ and integrate over the first quadrant:

$$\int_0^\infty \int_0^\infty f(\tau) g(u) \mathrm{e}^{-s(\tau+u)}\,\mathrm{d}u\,\mathrm{d}\tau = F(s)G(s).$$

Interchanging the order requires suitable absolute convergence.

Worked example

Example I:

Let $f(t) = \mathrm{e}^{-at}$ and $g(t) = \mathrm{e}^{-bt}$ with $a \neq b$. Then

$$(f * g)(t) = \int_0^t \mathrm{e}^{-a\tau}\mathrm{e}^{-b(t-\tau)}\,\mathrm{d}\tau = \frac{\mathrm{e}^{-at} - \mathrm{e}^{-bt}}{b-a}.$$

In the transform domain,

$$F(s)G(s) = \frac{1}{(s+a)(s+b)},$$

giving the same result after partial fractions.

Example II:

Since $1 \longleftrightarrow 1/s$,

$$(1 * 1)(t) = \int_0^t 1\,\mathrm{d}\tau = t \quad \longleftrightarrow \quad \frac{1}{s^2}.$$

Repeating the convolution,

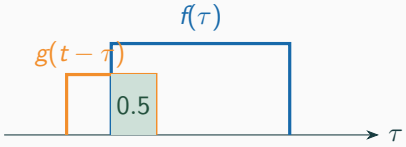
$$\underbrace{1 * \cdots * 1}_{n+1 \text{ factors}} = \frac{t^n}{n!} \quad \longleftrightarrow \quad \frac{1}{s^{n+1}}.$$

Convolution explains the factorial structure of the power transform pair.

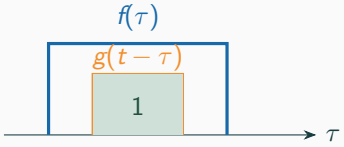
Convolution key frames

Let $f(\tau) = 1$ on $[0, 2]$ and $g(t - \tau) = 1$ on $[t - 1, t]$.

(t=0.5): partial overlap

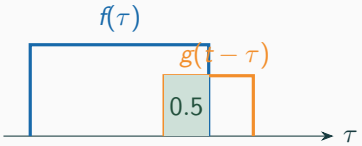


(t=1.5): full overlap of (g)

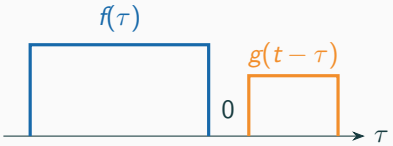


The overlap length grows from 0 to 1 as the shifted pulse enters.

(t=2.5): partial overlap



(t>3): no overlap

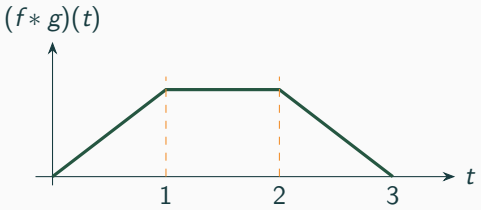


After the full-overlap interval, the accumulated overlap decreases to zero.

The overlap length is the convolution

For these two unit-height pulses,

$$(f * g)(t) = \begin{cases} t, & 0 \le t \le 1, \\ 1, & 1 \le t \le 2, \\ 3 - t, & 2 \le t \le 3, \\ 0, & \text{otherwise.} \end{cases}$$



What L11 does and does not do

Here convolution is a transform property and a calculation method. L14 will develop:

- impulse response and LTI systems;
- input–output interpretation;
- integral equations;
- convolution in PDE and system applications.

This boundary prevents repeating the same system theory twice.

A complete property workflow

1. rewrite piecewise signals with steps and delays;
2. identify scaling, shifts, derivatives, or integrals;
3. apply the minimum number of properties;
4. combine results and simplify singularities;
5. intersect and then recheck ROCs;
6. verify initial behavior or a direct special case.

Summary

- Linearity and scaling build transforms from basic pairs.
- Exponential weighting shifts s; delay introduces e^{-as}.
- Derivative rules include initial data; s-derivatives handle factors of t.
- Steps, impulses, and periodic formulas model engineering inputs.
- Time convolution becomes transform-domain multiplication.

Next $\gg$ Lecture 12 — Inverse Laplace Transform