L12 — Inverse Laplace Transform

Engineering Mathematics I

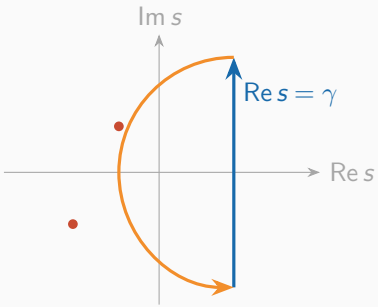
Profs. Tianyu DONG & Jiawei WANG

Fall 2026

School of Electrical Engineering, Xi'an Jiaotong University

Road map

- Meaning and uniqueness of the inverse transform
- Table-based and operational inversion
- Partial fraction decomposition
- Bromwich integral and contour inversion
- Residue inversion



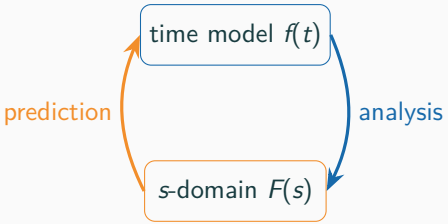
Forward analysis is only half the method

L10–L11 constructed

$$f(t) \xrightarrow{\mathcal{L}} F(s).$$

Engineering prediction requires the inverse direction

$$F(s) \xrightarrow{\mathcal{L}^{-1}} f(t).$$



Engineering question

Which inversion method exposes the time response with the least unnecessary work?

Meaning and Uniqueness of the Inverse Transform

Inverse Laplace transform

Suppose $F(s)$ is the Laplace transform of the function $f(t)$:

$$F(s) = \int_0^\infty f(t)\mathrm{e}^{-st}\,\mathrm{d}t,$$

then $f(t)$ is referred to as the inverse Laplace transform of the function $F(s)$:

$$\boxed{f(t) = \mathcal{L}^{-1}\{F(s)\}(t)}.$$

For piecewise-continuous functions of exponential order, the inverse is unique at continuity points in the causal sense. At a jump, the value can be redefined without changing the transform; in bilateral/Fourier inversion one often takes the midpoint.

Example

For $F(s) = \frac{1}{s}$,

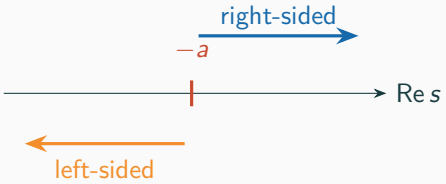
$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = u(t).$$

At $t = 0$, f jumps from 0 to 1; the midpoint value is $f(0) = \frac{1}{2}$.

The ROC remains part of the inverse problem

Bilaterally, the same algebraic expression has two inverses:

$$\frac{1}{s+a} \longleftrightarrow \begin{cases} \mathrm{e}^{-at}u(t), & \operatorname{Re} s > -a, \\ -\mathrm{e}^{-at}u(-t), & \operatorname{Re} s < -a. \end{cases}$$



Unilateral engineering problems normally select the causal first case.

Course convention

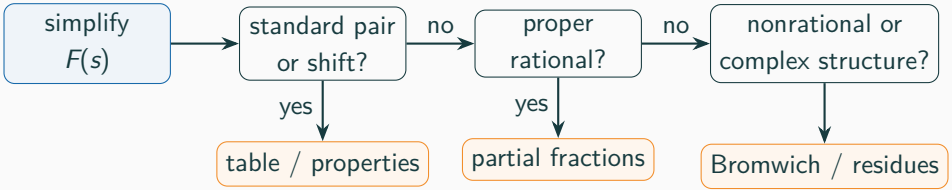
Unless otherwise stated, we use the unilateral causal inverse. All entries below are understood to include $u(t)$.

Four levels of inversion

Level	Main tools	Typical use
direct	transform table	standard pairs
operational	shifts, scaling, convolution	structured products
algebraic	partial fractions, cover-up	rational transforms
complex	Bromwich integral, residues	theory and difficult poles

Use the simplest level that preserves the required conditions and interpretation.

Inversion starts with classification



The classification determines the algebra before any calculation begins.

Table-Based and Operational Inversion

Read a transform pair in reverse

If

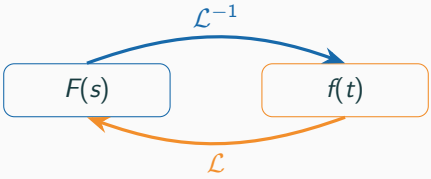
$$\mathcal{L}\{f(t)\} = F(s),$$

then the inverse is obtained by reading the table backwards:

$$\mathcal{L}^{-1}\{F(s)\} = f(t).$$

Match the algebraic form exactly: constants, shifts, and scaling must agree.

$F(s)$	$f(t)$
$1/s$	1
$1/s^{n+1}$	$t^n/n!$
$1/(s+a)$	e^{-at}
$\omega/(s^2+\omega^2)$	$\sin \omega t$
$s/(s^2+\omega^2)$	$\cos \omega t$



Worked Example

Example I:

$$F(s) = \frac{3}{s^2+4s+13} = \frac{3}{(s+2)^2+3^2}.$$

Therefore

$$\boxed{f(t) = \mathrm{e}^{-2t}\sin(3t)}.$$

The shift $s \mapsto s+2$ gives damping; the constant 3 gives the oscillation frequency.

Example II:

Consider

$$F(s) = \frac{2s+7}{s^2+6s+13} = \frac{2(s+3)+1}{(s+3)^2+2^2}$$
$$= \frac{2(s+3)}{(s+3)^2+2^2} + \frac{1}{2}\frac{2}{(s+3)^2+2^2}.$$

Hence

$$\boxed{f(t) = 2\mathrm{e}^{-3t}\cos 2t + \frac{1}{2}\mathrm{e}^{-3t}\sin 2t}.$$

Rewrite the numerator to match the shifted cosine and sine templates.

Time delay in inverse form

If

$$F(s) = \mathrm{e}^{-as} G(s),$$

then

$$\boxed{\mathcal{L}^{-1}\{F(s)\} = u(t-a)g(t-a)}.$$

The exponential factor in s indicates delayed onset, not exponential damping in time.

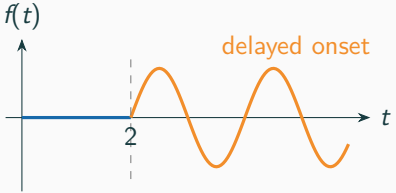
Worked example: delayed oscillation

$$F(s) = \mathrm{e}^{-2s}\frac{3}{s^2+9}.$$

Since $3/(s^2+9) \longleftrightarrow \sin 3t$,

$$\boxed{f(t) = u(t-2)\sin\big(3(t-2)\big)}.$$

The waveform uses the local clock $t-2$, not the original clock t.



Inverse scaling

From

$$f(at) \longleftrightarrow \frac{1}{a}F(s/a), \qquad a > 0,$$

obtain the inverse forms

$$\mathcal{L}^{-1}\{F(as)\} = \frac{1}{a}f(t/a)$$
$$\mathcal{L}^{-1}\{F(s/a)\} = af(at).$$

Check both the argument and the amplitude factor by substitution.

Common error

Time scaling changes both the transform argument and its amplitude factor.

Convolution as an inverse tool

If $F(s) = F_1(s)F_2(s)$, then

$$\mathcal{L}^{-1}\{F(s)\} = (f_1 * f_2)(t) = \int_0^t f_1(\tau) f_2(t-\tau)\,\mathrm{d}\tau.$$

Use convolution when the factors have simple inverses and the integral stays manageable. For most factorable rational functions, partial fractions are shorter.

Worked example: a product of transforms

$$F(s) = \frac{1}{s(s+a)} = \frac{1}{s}\frac{1}{s+a}.$$

Convolution gives

$$f(t) = \int_0^t 1 \cdot \mathrm{e}^{-a(t-\tau)}\,\mathrm{d}\tau = \boxed{\frac{1-\mathrm{e}^{-at}}{a}}.$$

This result can also be obtained by partial fractions. The product form makes convolution natural.

Partial Fraction Decomposition

Start with a proper rational function

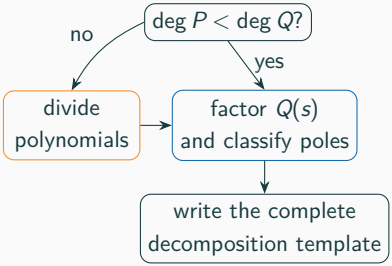
For

$$F(s) = \frac{P(s)}{Q(s)},$$

follow this preparation sequence:

1. Make the fraction proper: $\deg P < \deg Q$.
2. Factor $Q(s)$ as far as possible over the reals.
3. Record each factor and its multiplicity.
4. Write the full partial-fraction template before solving coefficients.

If the fraction is improper, divide polynomials first and keep any impulse terms separate.



Pole structure determines the algebraic template

Denominator factor	Required term	Time-domain mode
$s - p$	$A/(s-p)$	Ae^{pt}
$(s-p)^m$	$\sum_{k=1}^{m} A_k/(s-p)^k$	polynomial $\times e^{pt}$
$(s+a)^2 + \omega^2$	$[A(s+a)+B]/[(s+a)^2+\omega^2]$	damped sine/cosine

Before solving coefficients

The pole pattern already predicts decay rates, oscillation frequencies, and possible polynomial factors in t.

Distinct real poles

If

$$F(s) = \frac{P(s)}{(s-p_1)\cdots(s-p_n)},$$

with distinct poles, write

$$F(s) = \sum_{k=1}^{n} \frac{A_k}{s-p_k}, \qquad A_k = \lim_{s\to p_k} (s-p_k)F(s).$$

Each simple pole contributes one exponential mode $A_k \mathrm{e}^{p_k t}$.

Worked example: two distinct poles

$$F(s) = \frac{5s+7}{(s+1)(s+3)} = \frac{1}{s+1} + \frac{4}{s+3}.$$

Therefore

$$\boxed{f(t) = \mathrm{e}^{-t} + 4\mathrm{e}^{-3t}}.$$

Forward checking recombines the fractions to the original numerator.

Repeated poles

A pole of order m requires

$$\frac{A_1}{s-p} + \frac{A_2}{(s-p)^2} + \cdots + \frac{A_m}{(s-p)^m}.$$

1 → e^{pt}

2 → te^{pt}

3 → $\frac{1}{2}t^2e^{pt}$

Since

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-p)^k}\right\} = \frac{t^{k-1}}{(k-1)!}e^{pt},$$

every extra pole order raises the polynomial degree by one.

Worked example: a repeated pole

$$F(s) = \frac{s+4}{(s+2)^2} = \frac{1}{s+2} + \frac{2}{(s+2)^2}.$$

Hence

$$\boxed{f(t) = e^{-2t} + 2te^{-2t}}.$$

The factor t signals a second-order pole.

A repeated pole is visible in the complex domain

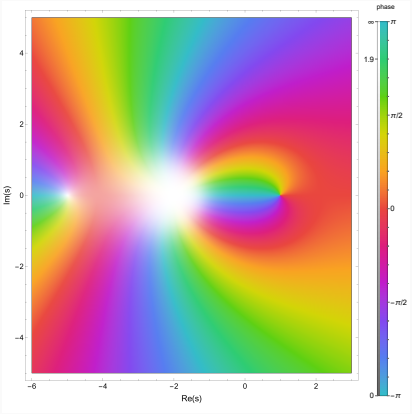
For

$$F(s) = \frac{s-1}{(s+2)^3(s+5)},$$

the triple pole at $s = -2$ dominates the local magnitude and phase pattern more strongly than the simple pole at $s = -5$.

Read the figure

Locate the singularities first; multiplicity determines the polynomial factor multiplying each exponential mode.



Irreducible quadratic factors

For

$$\frac{A(s+a)+B}{(s+a)^2+\omega^2},$$

the inverse is

$$\boxed{Ae^{-at}\cos(\omega t)+\frac{B}{\omega}e^{-at}\sin(\omega t)}.$$

The numerator must be linear because the denominator is quadratic.

Worked example: mixed pole types

Decompose

$$F(s)=\frac{2s+5}{(s+1)(s^2+4)}=\frac{3/5}{s+1}+\frac{-3s/5+13/5}{s^2+4}.$$

Thus

$$\boxed{f(t)=\frac{3}{5}e^{-t}-\frac{3}{5}\cos 2t+\frac{13}{10}\sin 2t}.$$

Improper rational transforms and impulses

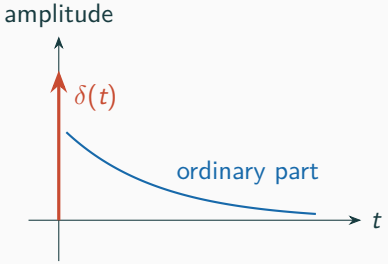
Polynomial division may produce

$$1, \quad s, \quad s^2.$$

In the generalized-function sense,

$$1 \longleftrightarrow \delta(t), \qquad s^n \longleftrightarrow \delta^{(n)}(t).$$

Separate impulse terms explicitly before inverting the proper remainder.



Partial-fraction workflow

1. make the rational function proper;
2. factor the denominator and record multiplicities;
3. write every required numerator term;
4. solve coefficients by limits or coefficient matching;
5. invert each standard term;
6. check initial value and pole-mode correspondence.

Bromwich Integral and Contour Inversion

The Bromwich formula is Fourier inversion on a vertical line

Set $s = \gamma + \mathrm{i}\omega$ in the forward transform:

$$F(\gamma + \mathrm{i}\omega) = \int_0^\infty \left[f(t)\mathrm{e}^{-\gamma t}\right]\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t.$$

Thus $F(\gamma + \mathrm{i}\omega)$ is the Fourier transform of the exponentially weighted signal. Fourier inversion gives

$$f(t) = \frac{\mathrm{e}^{\gamma t}}{2\pi}\int_{-\infty}^{\infty} F(\gamma + \mathrm{i}\omega)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega.$$

With $\mathrm{d}s = \mathrm{i}\,\mathrm{d}\omega$, this becomes the Bromwich integral.

Complex inversion formula

Under standard hypotheses,

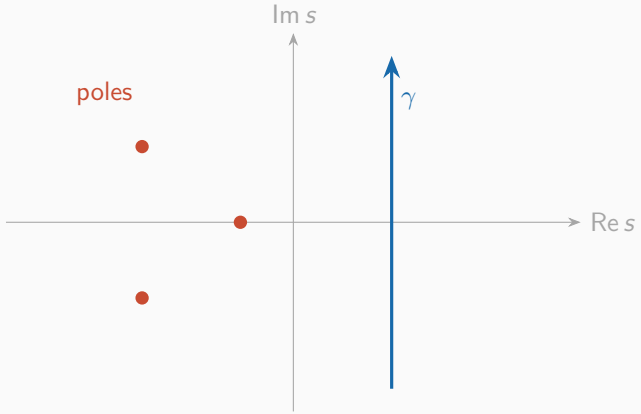
$$\boxed{f(t) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} F(s) \mathrm{e}^{st}\, \mathrm{d}s}.$$

The vertical line $\operatorname{Re} s = \gamma$ must lie inside the ROC and to the right of all singularities for a causal rational transform.

What γ does

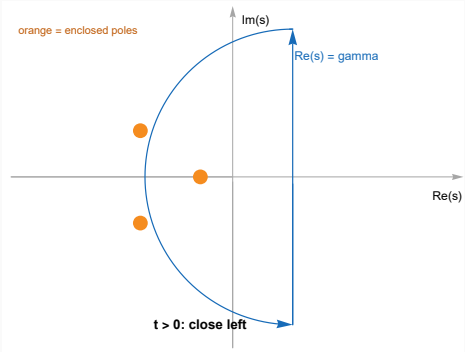
It supplies enough exponential damping for Fourier inversion; moving it inside the same ROC does not change the recovered signal.

Geometry of the Bromwich line

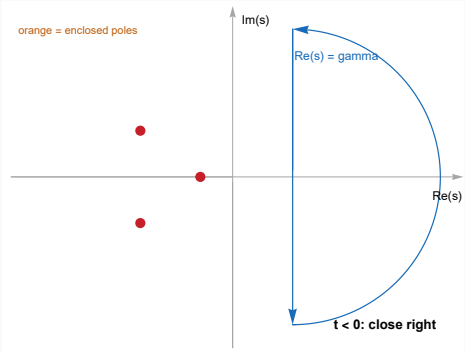


Moving the line within the same ROC does not change the inverse.

Contour key frames: the sign of time chooses the closure

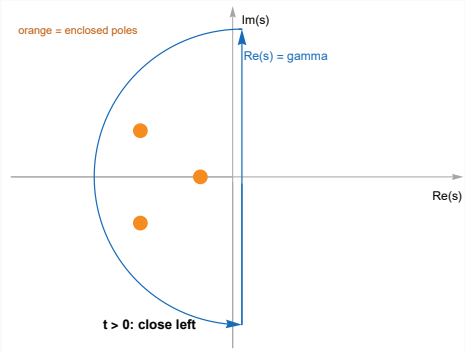


$t > 0$: close left, where e^{st} decays.

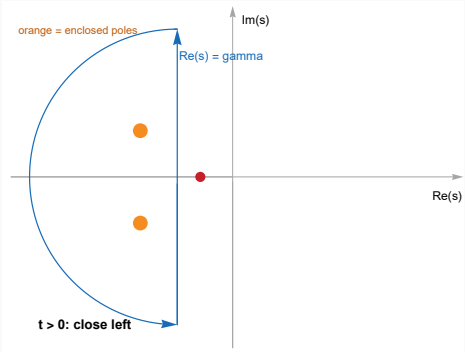


$t < 0$: close right; a causal transform encloses no poles.

Contour key frames: moving the inversion line



Moving γ without crossing a pole preserves the inverse.



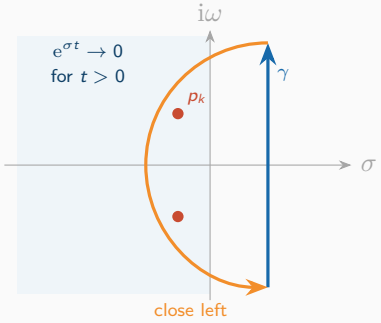
Crossing a pole changes the enclosed residue set and the ROC interpretation.

Why the contour closes to the left for $t > 0$

On a large closing arc, write $s = \sigma + \mathrm{i}\omega$. Then

$$\left|\mathrm{e}^{st}\right| = \left|\mathrm{e}^{\sigma t}\right| \underbrace{\left|\mathrm{e}^{\mathrm{i}\omega t}\right|}_{=1} = \mathrm{e}^{\sigma t}.$$

The arc contributes nothing only if $\mathrm{e}^{\sigma t} \to 0$ along it.



For $t > 0$:

- $\sigma \to -\infty$: $\mathrm{e}^{\sigma t} \to 0$ — left closure works;
- $\sigma \to +\infty$: $\mathrm{e}^{\sigma t} \to \infty$ — right closure fails.

Hence we close the Bromwich contour in the left half-plane.

Condition

The closure direction follows from exponential decay, but vanishing of the arc must still be checked for the particular transform.

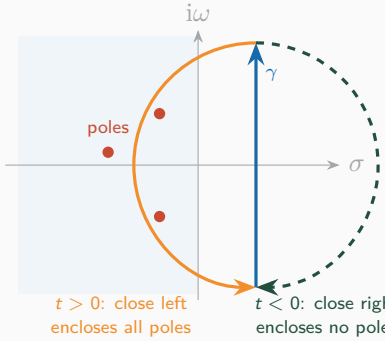
Causality appears through contour closure

For a causal unilateral transform, the ROC lies to the right of all poles. Choose γ in that ROC; then every pole lies to the left of γ.

- **For $t < 0$:** close right. The contour encloses no poles, hence
$$f(t) = 0.$$
- **For $t > 0$:** close left. The contour encloses all poles, giving
$$f(t) = \sum_k \operatorname{Res}\big(F(s)e^{st}; p_k\big)\,.$$

Therefore the causal inverse is automatically zero for $t < 0$:

$$\boxed{f(t) = f(t)u(t)}.$$

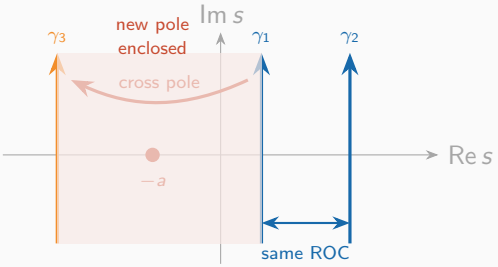


Interpretation

The contour argument supplies the factor $u(t)$ that is often left implicit in transform tables: a causal inverse simply *has no* left-sided part.

The inversion line cannot cross a pole

- **Within the same ROC:** moving γ crosses no pole, so the Bromwich integral is unchanged.
- **Crossing a pole:** the contour encloses an extra pole. Its residue adds to the inverse, so the time signal changes (the ROC changes).



Same ROC $\Rightarrow$ same inverse; crossing a pole $\Rightarrow$ different ROC, different inverse.

Example: $F(s) = \dfrac{1}{s+a}$, pole at $s = -a$.

$\gamma > -a:\ \mathrm{e}^{-at}u(t) \qquad \gamma < -a:\ -\mathrm{e}^{-at}u(-t)$

Rule

Place the Bromwich line in the stated ROC, not merely at any convenient positive real value.

Branch cuts require more than pole sums

Transforms containing s^{α}, $\log s$, or square roots may have branch points. Then the deformed contour includes:

- residues of isolated poles;
- integrals along the two sides of a branch cut;
- possible small-circle contributions near branch points.

Detailed branch-cut inversion is an advanced topic reserved for L16.

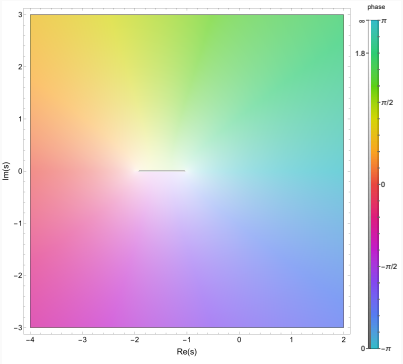
A logarithmic branch cut is an extended singular structure

For

$$F(s) = \log\left(\frac{s+1}{s+2}\right),$$

the branch points occur at $s = -1$ and $s = -2$. A chosen cut joins the discontinuous phase boundaries and makes one branch single-valued.

Reading cue: follow the abrupt phase change rather than searching for an isolated pole.



Poles and branch points require different inversions

Singular structure	Contour contribution	Typical method
simple or repeated poles	isolated residues	partial fractions / residues
logarithmic branch point	jump across a cut	Bromwich deformation
square-root branch point	two-sheet discontinuity	branch-cut integral / table

Read the phase discontinuity

A branch cut makes the function single-valued; its two sides contribute a jump integral rather than an isolated residue.

Extension example: a Bessel inverse

The standard pair

$$\boxed{\mathcal{L}\{J_0(at)\}(s) = \frac{1}{\sqrt{s^2 + a^2}}}, \qquad \operatorname{Re} s > 0,$$

gives

$$\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s^2 + a^2}}\right\} = J_0(at).$$

Unlike rational transforms, the square root has branch points at $s = \pm \mathrm{i}a$; partial fractions cannot be used.

Why this example belongs here

It marks the boundary between engineering table methods and full branch-cut inversion without developing special-function theory.

Square-root branch points control the Bessel inverse

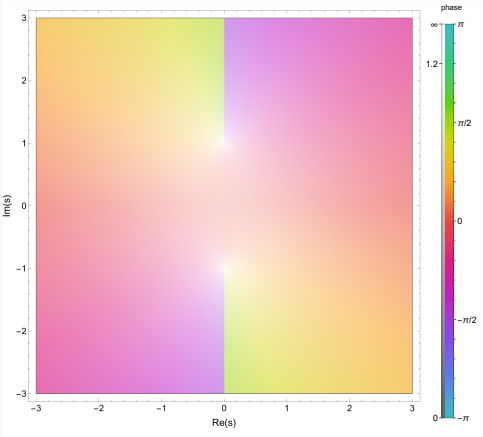
For

$$F(s) = \frac{1}{\sqrt{1+s^2}},$$

the branch points are $s = \pm \mathrm{i}$. The phase pattern reveals why no finite pole sum can recover the inverse.

Read the figure

The branch-cut jump replaces the isolated residue contribution.



Residue Inversion

From cover-up coefficients to residues

For a simple pole p_k, the partial-fraction coefficient is

$$A_k = \lim_{s \to p_k} (s - p_k) F(s).$$

The same limit is the residue of F at p_k:

$$A_k = \operatorname{Res}(F; p_k).$$

Thus partial fractions and residue inversion are two views of the same local pole information.

Bridge

Cover-up coefficients are the algebraic shadow of residues; residue inversion will use them directly in the time-domain formula.

Residue formula for causal rational transforms

When the left closure is valid and F is meromorphic,

$$\boxed{f(t) = \sum_{k} \operatorname{Res}\big(F(s)\mathrm{e}^{st}; p_k\big)}, \qquad t > 0,$$

where the sum includes poles enclosed to the left of the Bromwich line.

Interpretation

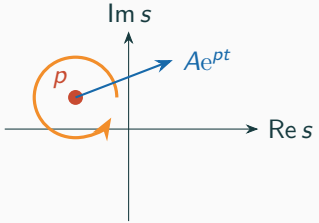
Complex inversion replaces one infinite contour integral by a finite sum of local pole contributions.

Simple-pole contribution

If F has a simple pole at p,

$$\operatorname{Res}\big(F(s)\mathrm{e}^{st};p\big) = \mathrm{e}^{pt}\lim_{s\to p}(s-p)F(s).$$

The pole location supplies the mode e^{pt}; the residue supplies its amplitude.

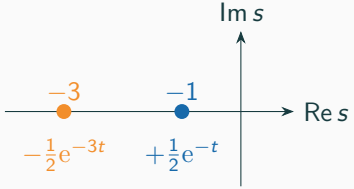


Worked example: residue inversion

$$F(s) = \frac{1}{(s+1)(s+3)}.$$

The residues at -1 and -3 give $\frac{1}{2}\mathrm{e}^{-t}$ and $-\frac{1}{2}\mathrm{e}^{-3t}$, so

$$\boxed{f(t) = \frac{1}{2}\big(\mathrm{e}^{-t} - \mathrm{e}^{-3t}\big)}.$$



Forward transformation recombines the two modes into the original rational function.

Repeated-pole residue

For a pole of order m at p,

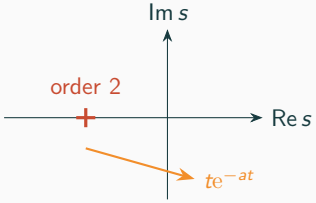
$$\operatorname{Res}(G;p)=\frac{1}{(m-1)!}\lim_{s\to p}\frac{\mathrm{d}^{m-1}}{\mathrm{d}s^{m-1}}\left[(s-p)^m G(s)\right].$$

Applied to $G(s)=F(s)\mathrm{e}^{st}$, derivatives of the exponential generate powers of t.
Consistency check: The highest power is t^{m-1}, exactly as predicted by the repeated-pole partial-fraction terms.

Worked example: second-order pole

For $F(s)=1/(s+a)^2$,

$$\operatorname{Res}\left(\frac{\mathrm{e}^{st}}{(s+a)^2};-a\right)=\left.\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{e}^{st}\right|_{s=-a}=t\mathrm{e}^{-at}.$$



Therefore

$$\boxed{\mathcal{L}^{-1}\{(s+a)^{-2}\}=t\mathrm{e}^{-at}}.$$

Complex-conjugate poles yield real signals

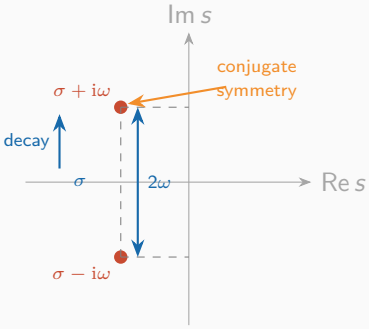
For a real-coefficient rational $F(s)$, poles and residues come in conjugate pairs:

$$p = \sigma + \mathrm{i}\omega, \quad \bar{p} = \sigma - \mathrm{i}\omega, \qquad A,\ \bar{A}.$$

Their time-domain contributions add:

$$\begin{aligned} A\mathrm{e}^{(\sigma+\mathrm{i}\omega)t} + \bar{A}\mathrm{e}^{(\sigma-\mathrm{i}\omega)t} &= \mathrm{e}^{\sigma t}\left(A\mathrm{e}^{\mathrm{i}\omega t} + \bar{A}\mathrm{e}^{-\mathrm{i}\omega t}\right) \\ &= 2\mathrm{e}^{\sigma t}\,\mathrm{Re}\left(A\mathrm{e}^{\mathrm{i}\omega t}\right) \\ &= 2|A|\mathrm{e}^{\sigma t}\cos(\omega t + \phi). \end{aligned}$$

The imaginary parts cancel: the result is a real damped sinusoid.



σ: damping rate ω: oscillation frequency

Key point

Real coefficients force pole/residue symmetry; adding the two conjugate modes makes the imaginary parts cancel, leaving a real signal.

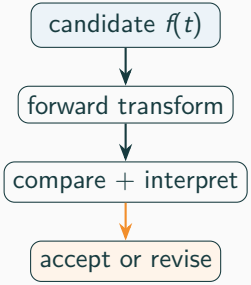
Which inversion method should we choose?

Transform structure	Preferred method
recognizable standard pair	table and shifts
rational with factorable denominator	partial fractions
convenient product	convolution
many poles or theoretical analysis	residues
branch point or nonrational form	Bromwich deformation

Verification closes the inversion loop

After obtaining a candidate $f(t)$:

1. transform it forward;
2. simplify and compare with $F(s)$;
3. check the ROC and causal support;
4. inspect $f(0^+)$ and large-time behavior;
5. compare poles with exponential modes.



Initial and final values as quick checks

Under the usual pole conditions,

$$f(0^+) = \lim_{s\to\infty} sF(s),$$

and

$$f(\infty) = \lim_{s\to 0} sF(s).$$

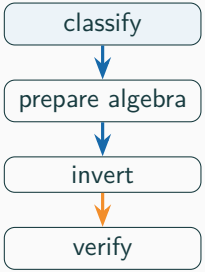
The final-value theorem requires all poles of $sF(s)$ to lie in the open left half-plane. Equivalently, $F(s)$ may have a simple pole at the origin, but no other pole on or to the right of the imaginary axis.

Limits are checks, not inversion formulas

The initial-value theorem excludes hidden impulse terms; the final-value theorem fails for sustained oscillation or unstable poles.

Complete inversion workflow

1. state the convention and ROC;
2. simplify and classify $F(s)$;
3. try table and operational properties;
4. use partial fractions for rational transforms;
5. use residues or Bromwich deformation only when needed;
6. verify by forward transform and qualitative checks.



Summary

- Inversion requires both an algebraic transform and its convergence interpretation.
- Tables, shifts, and partial fractions solve most engineering inversions.
- Repeated poles generate polynomial factors multiplying exponentials.
- The Bromwich integral provides the theoretical inverse.
- Residues convert pole information directly into time-domain modes.

Next $\gg$ Lecture 13 — Solving ODEs with Laplace Transform