L13 — Solving ODEs with Laplace Transform

Engineering Mathematics I

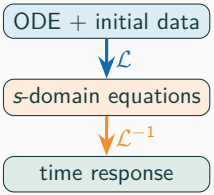
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Road map

- Method for solving initial-value problems
- Solution structure of constant-coefficient ODEs
- Discontinuous and impulsive forcing
- Systems of ODEs
- Applications to engineering problems
- A brief extension to polynomial coefficients



Ordinary Differential Equation (ODE)

From transform rules to a solution method

L10–L12 developed the transform, its operational rules, and inversion. We now combine them:

$$\boxed{\text{model} \longrightarrow \text{algebra in } s \longrightarrow Y(s) \longrightarrow y(t)}.$$

The advantage is structural: initial data enter during transformation, while steps and impulses remain ordinary algebraic factors.

Chapter boundary

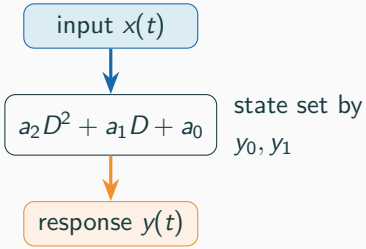
L13 solves ODE models. L14 will reinterpret the same algebra through linear time invariant (LTI) systems, impulse responses, convolution, and PDEs.

Method for Solving Initial-Value Problem

A standard engineering initial-value problem

Consider

$$a_2 y''(t) + a_1 y'(t) + a_0 y(t) = x(t),$$
$$y(0) = y_0, \qquad y'(0) = y_1.$$



- the operator describes the physical model;
- the initial data describe the stored state;
- $x(t)$ is the applied input and $y(t)$ is the measured output.

Objective

Determine the causal response for $t \geq 0$ and interpret its transient and forced components.

Derivative transforms carry the initial data

For a piecewise-continuous function of exponential order, the unilateral transform gives

$$\mathcal{L}\{y'\} = sY - y(0)$$
$$\mathcal{L}\{y''\} = s^2 Y - sy(0) - y'(0).$$

More generally,

$$\mathcal{L}\{y^{(n)}\} = s^n Y - \sum_{k=0}^{n-1} s^{n-1-k} y^{(k)}(0^+).$$

Every derivative contributes one known initial-value term. Hence the transformed equation contains the complete initial state and no arbitrary constants need to be fitted after inversion.

From ODE to algebraic equation

Start from the standard second-order initial-value problem:

$$a_2 y'' + a_1 y' + a_0 y = x(t), \qquad y(0) = y_0, \quad y'(0) = y_1.$$

1. Transform each derivative.

$$\mathcal{L}\{y''\} = s^2 Y - s y_0 - y_1, \qquad \mathcal{L}\{y'\} = sY - y_0.$$

2. Substitute into the ODE.

$$a_2(s^2 Y - s y_0 - y_1) + a_1(sY - y_0) + a_0 Y = X(s).$$

3. Collect the $Y(s)$ terms.

$$\left(a_2 s^2 + a_1 s + a_0\right) Y = X(s) + \underbrace{a_2(s y_0 + y_1) + a_1 y_0}_{\text{initial-state contribution}}.$$

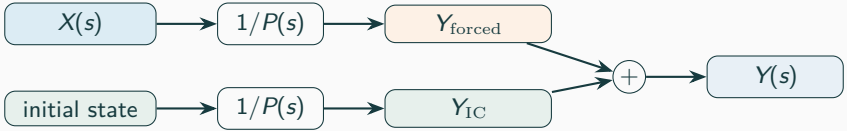
Decomposing $Y(s)$ as forced and initial-state parts

Define the characteristic denominator $P(s) = a_2 s^2 + a_1 s + a_0$. Then solving the transformed equation gives

$$Y(s) = \underbrace{\frac{X(s)}{P(s)}}_{Y_{\text{forced}}} + \underbrace{\frac{a_2(sy_0 + y_1) + a_1 y_0}{P(s)}}_{Y_{\text{IC}}}.$$

Two contributions

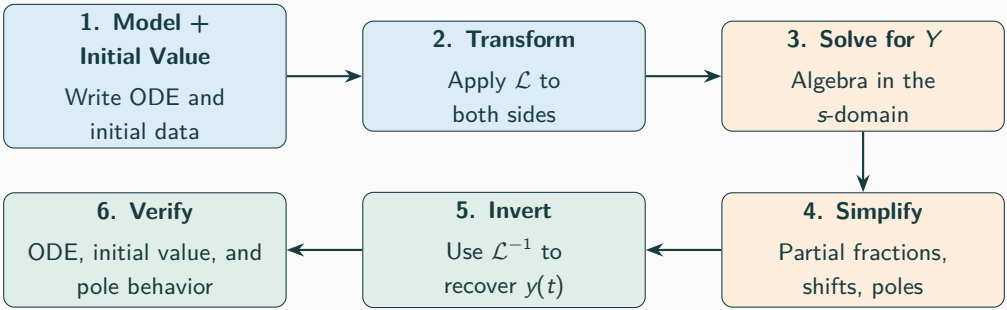
- Y_{forced}: response driven by the input $x(t)$.
- Y_{IC}: zero-input response produced by the initial state.



Key point

Both the input and the initial conditions pass through the *same* denominator $P(s)$. They share the same natural modes; only the numerators differ.

Workflow for solving an initial-value problem



Key idea

Laplace transform turns differentiation in t into multiplication by s; initial conditions enter automatically as known terms. After solving in the s-domain, invert and verify.

Worked example: transform the initial-value problem

Solve

$$y'' + 3y' + 2y = u(t-4), \qquad y(0) = 1, \quad y'(0) = -1.$$

1. Transform each term.

$$\mathcal{L}\{y''\} = s^2 Y - s y(0) - y'(0) = s^2 Y - s + 1,$$

$$\mathcal{L}\{y'\} = sY - y(0) = sY - 1, \qquad \mathcal{L}\{u(t-4)\} = \frac{\mathrm{e}^{-4s}}{s}.$$

2. Substitute into the ODE.

$$(s^2 Y - s + 1) + 3(sY - 1) + 2Y = \frac{\mathrm{e}^{-4s}}{s}.$$

3. Collect the $Y(s)$ terms.

$$\boxed{(s^2 + 3s + 2)Y = s + 2 + \frac{\mathrm{e}^{-4s}}{s}}.$$

Worked example: solve for $Y(s)$

Factor the characteristic denominator:

$$s^2 + 3s + 2 = (s+1)(s+2).$$

Hence

$$Y(s) = \frac{s+2}{(s+1)(s+2)} + \frac{\mathrm{e}^{-4s}}{s(s+1)(s+2)}.$$

Therefore

$$\boxed{Y(s) = \frac{1}{s+1} + \mathrm{e}^{-4s}\frac{1}{s(s+1)(s+2)}}.$$

Interpretation

- $\dfrac{1}{s+1}$: zero-input response from $y(0) = 1,\ y'(0) = -1$.
- $\mathrm{e}^{-4s}\dfrac{1}{s(s+1)(s+2)}$: delayed forced response caused by $u(t-4)$.

Worked example: partial fractions and inverse transform

Decompose the forced factor:

$$\frac{1}{s(s+1)(s+2)} = \frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2(s+2)}.$$

Substituting back,

$$Y(s) = \frac{1}{s+1} + \mathrm{e}^{-4s}\left(\frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2(s+2)}\right).$$

Therefore

$$\boxed{y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathrm{e}^{-t}u(t) + u(t-4)\left[\frac{1}{2} - \mathrm{e}^{-(t-4)} + \frac{1}{2}\mathrm{e}^{-2(t-4)}\right]}.$$

Equivalently,

$$y(t) = \begin{cases} \mathrm{e}^{-t}, & 0 \le t < 4, \\ \mathrm{e}^{-t} + \dfrac{1}{2} - \mathrm{e}^{-(t-4)} + \dfrac{1}{2}\mathrm{e}^{-2(t-4)}, & t \ge 4. \end{cases}$$

Worked example: verification

Initial conditions.

$$y(0^+) = \mathrm{e}^0 = 1, \qquad y'(0^+) = -\mathrm{e}^0 = -1.$$

Differential equation. For $t > 4$, the terms $\mathrm{e}^{-t}, \mathrm{e}^{-2t}$ solve the homogeneous equation, and the constant $1/2$ satisfies $2y = 1$, which is the forced equilibrium.

Final value.

$$\lim_{t\to\infty} y(t) = \frac{1}{2},$$

matching the static equilibrium $2y = 1$ when the step input is active.

Pole check

Poles at $s = -1, -2$ give the decaying exponentials. The simple pole at $s = 0$ from the step input produces the constant $1/2$.

Two equivalent routes: direct inversion or Green function

With zero initial data, define

$$G(s) = \frac{1}{P(s)}, \qquad g(t) = \mathcal{L}^{-1}\{G(s)\}.$$

Then

$$Y(s) = G(s)X(s) \quad \Longleftrightarrow \quad y(t) = \int_0^t g(t-\tau)x(\tau)\,\mathrm{d}\tau.$$

Two equivalent routes

- **Direct inversion:** compute $X(s)/P(s)$ and invert.
- **Green-function route:** first find $g(t)$, then convolve with $x(t)$.

Key point

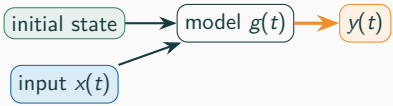
The Green-function route separates the *model* (described by $g(t)$) from the *input* $x(t)$. The same $g(t)$ can be reused for many different inputs. L14 will develop this convolution viewpoint as system theory.

Initial response plus forced response

For general initial data,

$$Y(s) = Y_{\mathrm{IC}}(s) + G(s)X(s), \qquad \boxed{y(t) = y_{\mathrm{IC}}(t) + (g * x)(t)}.$$

- y_{IC} reveals stored-energy decay;
- $(g * x)$ reveals how the model filters the input;
- the same g can be reused for many inputs.



Boundary data require an additional algebraic step

For example,

$$y'' + 9y = \cos 2t, \qquad y(0) = 1, \quad y(\pi/2) = -1.$$

Treat $y'(0)$ as an unknown, invert the transformed expression, and use the second boundary condition to determine it.

Method choice

Laplace transforms are naturally aligned with initial data at one endpoint. For general boundary-value problems, other methods may be more direct.

Solution Structure of Constant-Coefficient ODEs

The characteristic denominator organizes the solution

For $P(s) = a_2 s^2 + a_1 s + a_0$, its roots determine the natural modes:

Pole pattern	Time-domain mode
real pole p	e^{pt}
repeated pole p	$t^k \mathrm{e}^{pt}$
pair $\alpha \pm \mathrm{i}\beta$	$\mathrm{e}^{\alpha t}(A\cos\beta t + B\sin\beta t)$

Read the poles before inverting

Negative real parts give decay, positive real parts give growth, and nonzero imaginary parts give oscillation. Repeated poles add polynomial factors in t.

Worked example: delayed forcing

Consider

$$y'' + 2y = 5u(t-1), \qquad y(0) = 0, \quad y'(0) = 1.$$

Transformation gives

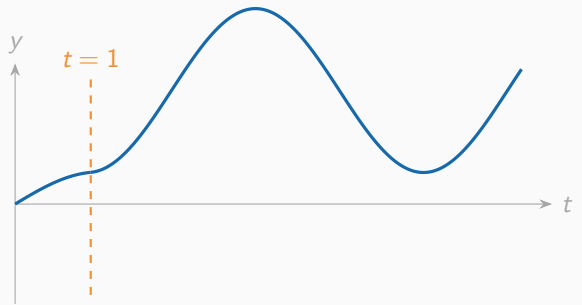
$$(s^2+2)Y(s) - 1 = \frac{5\mathrm{e}^{-s}}{s}, \qquad Y(s) = \frac{1}{s^2+2} + \frac{5\mathrm{e}^{-s}}{s(s^2+2)} = \frac{1}{s^2+2} + \frac{5}{2}\mathrm{e}^{-s}\left(\frac{1}{s} - \frac{s}{s^2+2}\right).$$

The denominator s^2+2 has poles $s = \pm \mathrm{i}\sqrt{2}$, predicting an oscillation at frequency $\sqrt{2}$. The additional pole at $s = 0$ comes from the step input and will produce a constant term.

$$\boxed{y(t) = \frac{\sin(\sqrt{2}, t)}{\sqrt{2}} + \frac{5}{2}u(t-1)\big[1 - \cos\big(\sqrt{2}(t-1)\big)\big]}.$$

Delayed forcing: final response

$$y(t) = \frac{\sin(\sqrt{2}, t)}{\sqrt{2}} + \frac{5}{2}u(t-1)\left[1 - \cos\left(\sqrt{2}(t-1)\right)\right].$$



The poles $s = \pm i\sqrt{2}$ give the oscillation; the pole at $s = 0$ from the step gives the constant $5/2$ after the delay.

Forced oscillation and resonance

For the zero-state problem

$$x'' + \omega_0^2 x = F_0 \cos \omega t, \qquad x(0) = x'(0) = 0,$$

the transformed response is

$$X(s) = \frac{sF_0}{(s^2 + \omega^2)(s^2 + \omega_0^2)}.$$

If $\omega \neq \omega_0$, partial fractions give

$$x(t) = \frac{F_0}{\omega_0^2 - \omega^2} \left(\cos \omega t - \cos \omega_0 t \right),$$

a bounded superposition of the forcing and natural frequencies.

At resonance the repeated poles change the response class

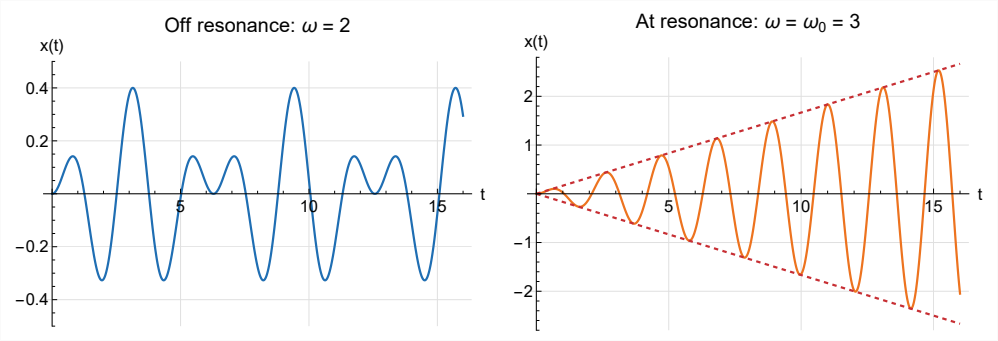
When $\omega = \omega_0$, the two quadratic factors merge:

$$X(s) = \frac{sF_0}{(s^2 + \omega_0^2)^2}$$

$$\boxed{x(t) = \frac{F_0}{2\omega_0} t \sin \omega_0 t}.$$

- the oscillation frequency remains ω_0;
- the envelope grows linearly because the poles are repeated;
- damping, nonlinearities, and finite energy prevent unlimited growth in real systems.

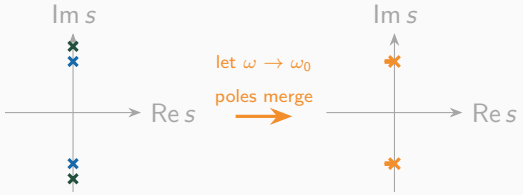
Comparison: off resonance and at resonance



Distinct frequencies produce beats; coincident frequencies produce a linearly growing envelope.

Resonance is visible in the pole structure

Same location, different multiplicity $\Longrightarrow$ different time-domain behavior.



Off resonance $\omega \neq \omega_0$

four distinct poles

At resonance $\omega = \omega_0$

two *double* poles

Pole pattern

Situation	Pole structure
off resonance	$(s^2+\omega^2)(s^2+\omega_0^2)$
at resonance	$(s^2+\omega_0^2)^2$

Read the response from the poles

- distinct poles $\Rightarrow$ bounded superposition of two frequencies;
- *double* poles $\Rightarrow$ an extra factor t in the time response

Discontinuous and Impulsive Forcing

Three idealized inputs: step, pulse, and impulse



step



finite pulse



impulse

Step

$u(t-a)$

Switch closed; persistent action.

$$\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}$$

Finite pulse

$A[u(t-a) - u(t-b)]$

Short voltage or force pulse.

$$\mathcal{L}\{p(t)\} = \frac{A}{s}(e^{-as} - e^{-bs})$$

Impulse

$J\delta(t-a)$

Hammer blow; instantaneous charging.

$$\mathcal{L}\{J\delta(t-a)\} = Je^{-as}$$

Key idea

Step: persistent action; **pulse:** finite-duration action; **impulse:** instantaneous action.

Their Laplace transforms are simple algebraic factors, which makes them ideal for solving ODEs.

Switching is encoded by a time shift

Suppose a signal $g(t)$ is switched on at time $t = a$:

$$f(t) = \begin{cases} 0, & t < a, \\ g(t-a), & t \geq a. \end{cases}$$

Using the unit step, $\boxed{f(t) = u(t-a)\,g(t-a)}$.

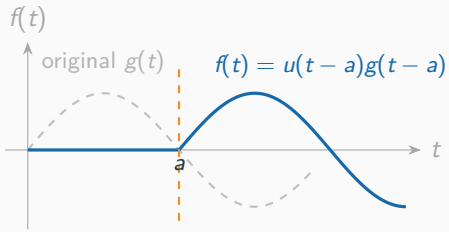
Why $t - a$? The waveform starts at $t = a$, so inside the switched-on region we use the *local clock*

$$\tau = t - a \geq 0.$$

Thus $g(t-a)$ is exactly the original shape $g(\tau)$ shifted to start at a.

Transform: $\boxed{\mathcal{L}\{u(t-a)g(t-a)\} = \mathrm{e}^{-as}G(s)}$.

The time delay becomes a simple exponential factor e^{-as}.



Key point

The delay a appears in two places:

- as the switch $u(t-a)$;
- as the local time $t - a$ inside g.

A pulse is the difference of two steps

A rectangular pulse can be built from **two steps**:

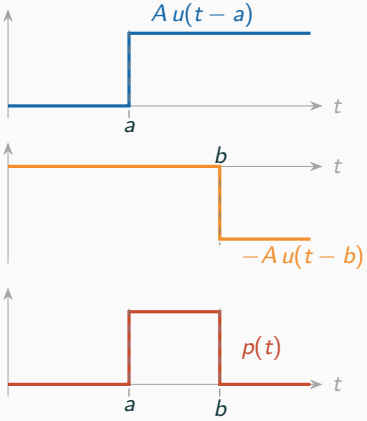
- one turns **on** at $t = a$;
- one turns **off** at $t = b$.

Therefore

$$\boxed{p(t) = A\big[u(t-a) - u(t-b)\big]}.$$

Its transform is just the difference of two delayed step transforms:

$$\boxed{P(s) = \frac{A}{s}\left(\mathrm{e}^{-as} - \mathrm{e}^{-bs}\right)}.$$



Why this is useful

We already know the step transform and the step response. By linearity, the pulse response is the step response at a minus the step response at b.

An impulse creates a jump in velocity, not in position

Consider

$$my'' + cy' + ky = J\delta(t-a).$$

Integrate across $t = a$ and then

$$m\big[y'(a^+) - y'(a^-)\big] = J.$$

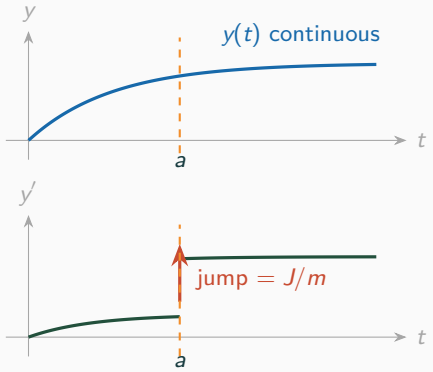
where $a^{\pm}$ denote just after / just before the impulse.

- **Position $y(t)$: continuous at a.**
- **Velocity $y'(t)$: jumps by J/m.**

So an impulse sets a *jump condition*, not a finite force value.

Transform: $\boxed{\mathcal{L}\{J\delta(t-a)\} = Je^{-as}}$.

The factor e^{-as} records *when* the blow occurs.



Key point

Impulse $\delta(t-a) \longrightarrow$ jump in y' at $t = a$.

Worked example: impulsively forced oscillator

Problem:

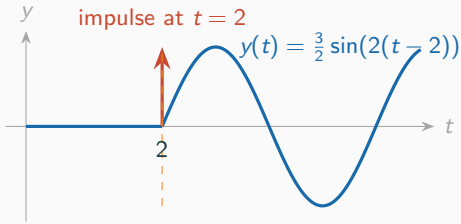
$$y'' + 4y = 3\delta(t-2), \quad y(0) = y'(0) = 0.$$

Transform:

$$s^2 Y + 4Y = 3e^{-2s} \Longrightarrow Y(s) = \frac{3e^{-2s}}{s^2+4}.$$

Invert:

$$\boxed{y(t) = \frac{3}{2}\, u(t-2)\sin\big(2(t-2)\big)}.$$



Key point

The impulse at $t = 2$ starts the oscillation. The response is zero before the impulse and sinusoidal after.

Verify: Before $t = 2$, $y(t) = 0$. At $t = 2$,

$$y(2^+) = y(2^-) = 0, \qquad y'(2^+) - y'(2^-) = 3,$$

exactly matching the impulse strength.

Choose the representation before transforming

Input feature	Time-domain form	Transform feature
switch at a	$u(t-a)g(t-a)$	$\mathrm{e}^{-as}G(s)$
finite pulse	difference of steps	difference of delays
instantaneous input	$J\delta(t-a)$	$J\mathrm{e}^{-as}$
periodic input	one-period formula	geometric denominator

Key idea

Choosing the correct representation is part of modelling, not a later algebraic detail. Once the input is written in transform-ready form, the ODE becomes ordinary algebra in s.

Systems of ODEs

A vector initial-value problem becomes matrix algebra

For

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}, \qquad \mathbf{x}(0) = \mathbf{x}_0,$$

where $\mathbf{x}$ is a unknown vector and A is a matrix.

Then, transformation gives

$$s\mathbf{X}(s) - \mathbf{x}_0 = A\mathbf{X}(s) + \mathbf{F}(s).$$

Therefore

$$\boxed{\mathbf{X}(s) = (sI - A)^{-1}\big(\mathbf{x}_0 + \mathbf{F}(s)\big)}.$$

The poles are the eigenvalues of A unless pole–zero cancellation occurs.

Worked example: a coupled first-order system

Solve

$$x' = x + y, \qquad y' = -2x, \qquad x(0) = 1, \quad y(0) = 0.$$

Transformation produces

$$(s-1)X - Y = 1, \qquad 2X + sY = 0.$$

Thus

$$X(s) = \frac{s}{s^2 - s + 2}, \qquad Y(s) = \frac{-2}{s^2 - s + 2}.$$

Since

$$s^2 - s + 2 = \left(s - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{7}}{2}\right)^2,$$

$$x(t) = \mathrm{e}^{t/2}\left[\cos\frac{\sqrt{7}t}{2} + \frac{1}{\sqrt{7}}\sin\frac{\sqrt{7}t}{2}\right], \qquad y(t) = -\frac{4}{\sqrt{7}}\mathrm{e}^{t/2}\sin\frac{\sqrt{7}t}{2}.$$

The positive real part $1/2$ predicts growing oscillations.

The shared poles govern both state variables

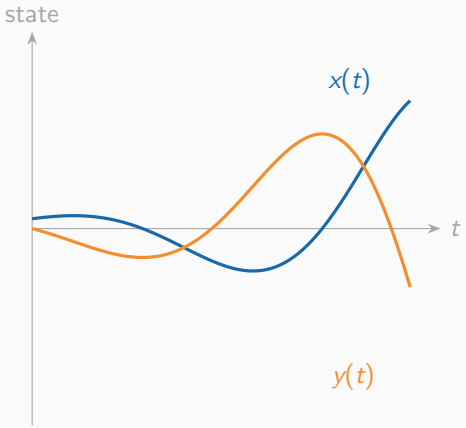
Both $X(s)$ and $Y(s)$ contain

$$s^2 - s + 2,$$

so both states have the same growth rate and oscillation frequency. Their numerators determine amplitude and phase.

Interpret before plotting

The factor $e^{t/2}$ signals instability even though the trajectories oscillate.



Normal modes simplify coupled oscillators

For two identical coupled coordinates,

$$m\mathbf{x}'' + K\mathbf{x} = 0, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

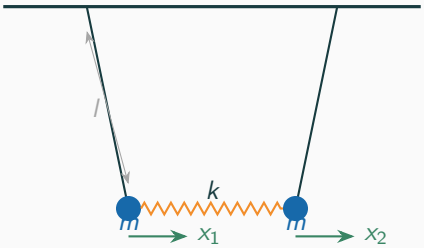
symmetric and antisymmetric coordinates

$$q_+ = x_1 + x_2, \qquad q_- = x_1 - x_2$$

often decouple the equations. In the transform domain this diagonalizes the matrix polynomial $ms^2 I + K$.

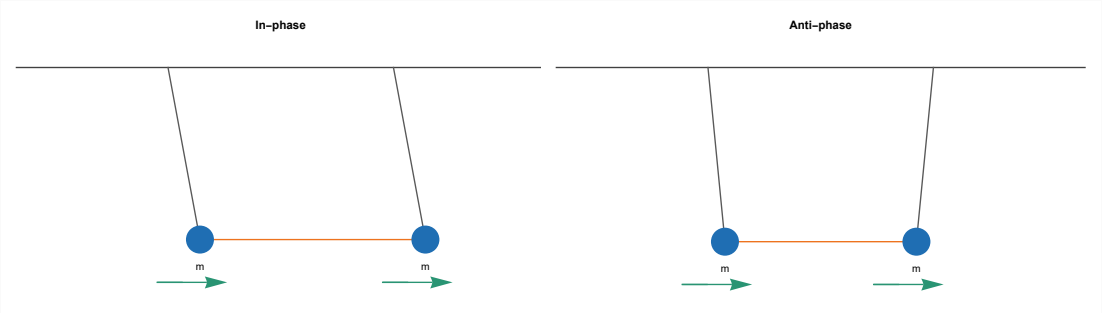
Mode	Relation	Physical motion
symmetric	$x_1 = x_2$	masses move together
antisymmetric	$x_1 = -x_2$	masses move oppositely

Coupled pendulums: the model geometry



Transformation converts the coupled ODEs into simultaneous algebraic equations for $X_1(s)$ and $X_2(s)$.

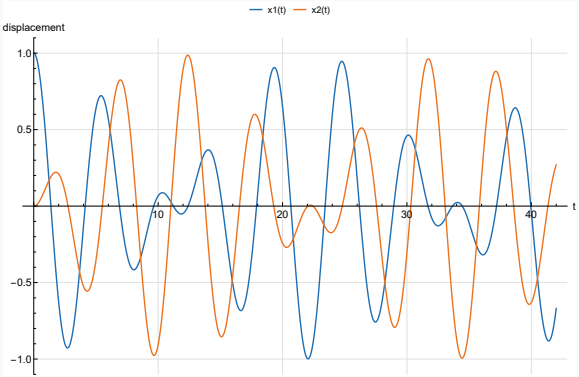
Normal-mode key frames



Symmetric mode: equal displacement.

Antisymmetric mode: opposite displacement.

A general initial condition exchanges energy between modes



The two modal frequencies interfere, producing a slow exchange envelope between $x_1(t)$ and $x_2(t)$.

A reliable workflow for coupled systems

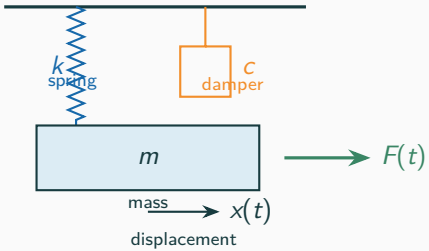
1. write all equations with one sign convention;
2. transform the entire system and insert every initial condition;
3. solve the matrix algebra before partial fractions;
4. factor the common determinant and identify modal poles;
5. invert each component and verify all initial values.

Common error

Omitting one component's initial derivative changes the right-hand side of the whole algebraic system.

Applications to Engineering Problems

Mass–spring–damper: from physics to the denominator



Newton's second law: $mx'' + cx' + kx = F(t)$

$$x'' + 2\alpha x' + \omega_n^2 x = F(t)/m$$

where $2\alpha = c/m$ and $\omega_n^2 = k/m$.

Transform with zero initial data:

$$(s^2 + 2\alpha s + \omega_n^2)X(s) = F(s)/m.$$

The denominator $\boxed{s^2 + 2\alpha s + \omega_n^2}$ is the *characteristic polynomial*. Its roots determine whether the system decays, oscillates, or does both.

Physical meaning

Symbol	Role
m	inertia (mass)
c	dissipation (damping)
k	restoring force (spring)
$F(t)$	external input
$x(t)$	output displacement

Interpretation of parameters

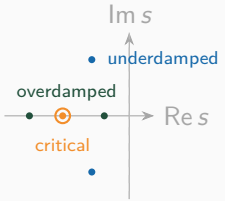
- α: damping rate (how fast energy is lost).
- ω_n: natural frequency (oscillation without damping).

Damping classes are pole configurations

Characteristic equation:

$$s^2 + 2\alpha s + \omega_n^2 = 0 \implies s = -\alpha \pm \sqrt{\alpha^2 - \omega_n^2}, \qquad \omega_d = \sqrt{\omega_n^2 - \alpha^2}.$$

Condition	Poles	Response
$\alpha < \omega_n$	$-\alpha \pm \mathrm{i}\omega_d$	decaying oscillation
$\alpha = \omega_n$	repeated $-\alpha$	fastest nonoscillatory
$\alpha > \omega_n$	two negative real	slower nonoscillatory



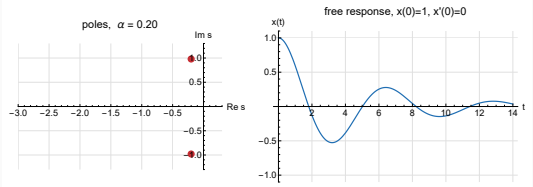
Read the table

- Complex poles ⇒ oscillation.
- Real poles ⇒ no oscillation.
- Repeated pole ⇒ boundary case.
- α: decay rate; ω_d: damped frequency.

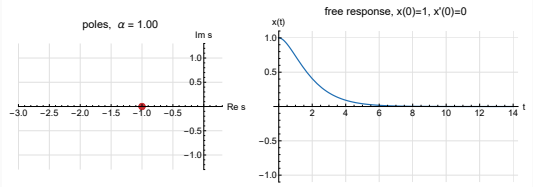
One continuous picture

As α grows, the complex pair moves toward the real axis, merges at $\alpha = \omega_n$, then splits into two real poles.

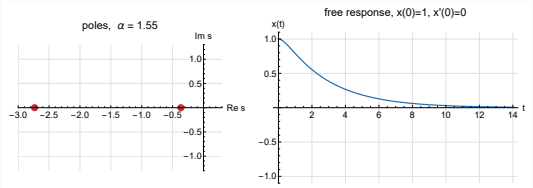
Damping changes both poles and time response



underdamped



critical



overdamped

Pole geometry and time response change together as α crosses ω_n.

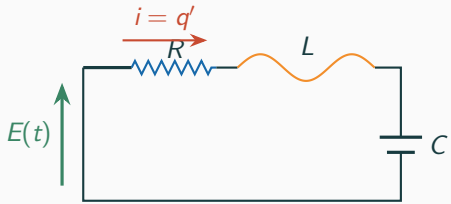
What to observe in the damping demonstration

As the damping ratio increases:

- complex poles move toward the negative real axis;
- oscillation frequency decreases in the underdamped regime;
- the poles merge at critical damping;
- beyond critical damping they split into two real negative poles.

Critical damping gives the fastest return without oscillation; excessive damping can slow the dominant mode.

A series RLC circuit



$q(0) = q_0, \quad q'(0) = i_0$

Kirchhoff's voltage law:

$$Lq'' + Rq' + \frac{1}{C}q = E(t)$$

Transform:

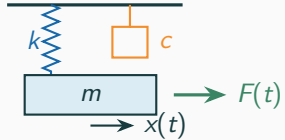
$$\left(Ls^2 + Rs + \frac{1}{C}\right)Q = E(s) + L(sq_0 + i_0) + Rq_0.$$

$$\boxed{Q(s) = \frac{1}{Ls^2 + Rs + 1/C}\Big(E(s) + L(sq_0 + i_0) + Rq_0\Big)}$$

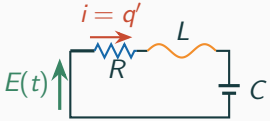
Key point

The denominator $P(s) = Ls^2 + Rs + 1/C$ is the characteristic polynomial. Its roots determine damping and oscillation—exactly as in the mechanical mass–spring–damper system.

Mechanical and electrical oscillators share one equation



Mechanical system



Series RLC circuit

$$mx'' + cx' + kx = F(t)$$

$$Lq'' + Rq' + q/C = E(t)$$

Mechanical	Electrical	Mathematical role
m	L	inertia
c	R	dissipation
k	$1/C$	restoring effect
$F(t)$	$E(t)$	external input

Worked example: impulsively driven RLC circuit

Problem:

$$50q'' + 3q' + 0.1q = \delta(t), \qquad q(0) = q'(0) = 0.$$

Transform:

$$Q(s) = \frac{1}{50s^2 + 3s + 0.1} = \frac{1/50}{(s + 0.03)^2 + (\sqrt{11}/100)^2}.$$

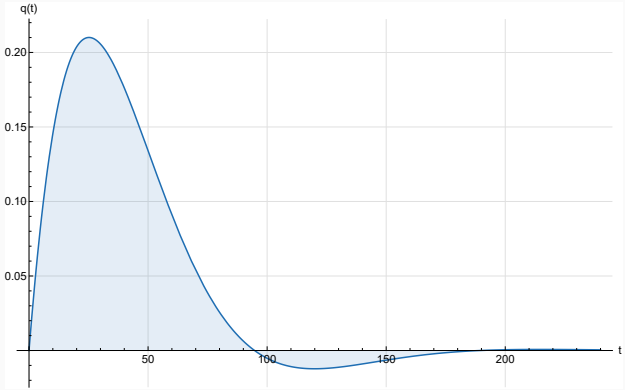
Invert:

$$\boxed{q(t) = \frac{2}{\sqrt{11}} \mathrm{e}^{-0.03t} \sin\left(\frac{\sqrt{11}}{100} t\right).}$$

Physical reading

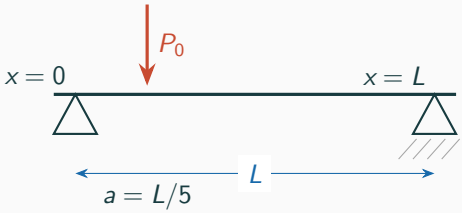
- $\delta(t)$ gives an instantaneous energy kick at $t = 0$.
- The pole pair $s = -0.03 \pm \mathrm{i}\frac{\sqrt{11}}{100}$ produces a damped oscillation.
- Decay rate: 0.03.
- Damped frequency: $\frac{\sqrt{11}}{100} \approx 0.0332$.

The RLC impulse produces a damped charge response



The exponential envelope comes from $\operatorname{Re} p = -0.03$; the pole pair supplies the oscillation.

A beam problem uses the same transform logic



For a simply supported beam with a point load,

$$EIy^{(4)}(x) = P_0\delta(x-a).$$

The boundary conditions are

$$y(0) = y(L) = 0, \qquad y''(0) = y''(L) = 0.$$

Here the transform variable is paired with x, not time.

Transform first, then enforce the far-end conditions

Transforming from $x = 0$ gives

$$EI\left[s^4 Y - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0)\right] = P_0 \mathrm{e}^{-as}.$$

The left-end conditions remove $y(0)$ and $y''(0)$; the values $y'(0)$ and $y'''(0)$ remain unknown until the right-end conditions are applied after inversion.

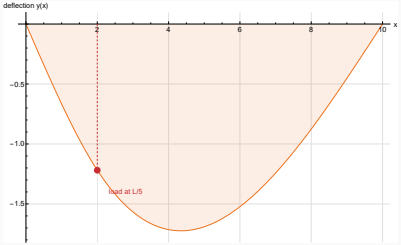
Why this example belongs here

It demonstrates the same transform logic with a spatial variable, while detailed beam theory and general boundary-value methods remain outside this course.

The endpoint conditions determine the deflection curve

Let $b = L - a$. For the simply supported beam,

$$EI\,y(x) = \begin{cases} -\dfrac{P_0 b x}{6L}\left(L^2 - b^2 - x^2\right), & 0 \le x \le a, \\ -\dfrac{P_0 a (L-x)}{6L}\left(L^2 - a^2 - (L-x)^2\right), & a \le x \le L. \end{cases}$$



A Brief Extension to Polynomial Coefficients

Multiplication by time becomes differentiation in s

$$\boxed{\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}s^n} F(s)}.$$

Terms such as $t\,y'(t)$ or $t\,y''(t)$ introduce derivatives of $Y(s)$. For example,

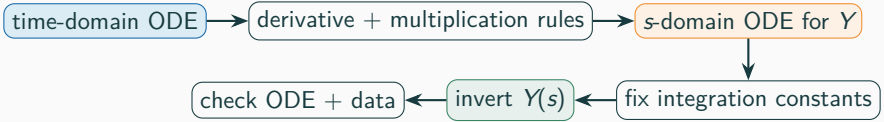
$$\mathcal{L}\{t\,y'(t)\} = -\frac{\mathrm{d}}{\mathrm{d}s}\big[sY(s) - y(0)\big].$$

A variable-coefficient ODE in time generally becomes an ODE in s rather than an algebraic equation.

How the problem changes in the transform domain

Time-domain model	Transform-domain problem	Main task
constant coefficients	algebraic equation for Y	factor and invert
polynomial coefficients	differential equation for Y	solve, fix constants, invert

Initial- and final-value information may be needed to determine integration constants in the s-domain solution.



Compared with the constant-coefficient workflow, the new task is solving a differential equation in the transform variable.

Extension example

Consider

$$ty'' + (t-1)y' - y = 0, \qquad y(0) = 3, \quad y(t) \to 0.$$

Using the derivative and multiplication rules gives

$$s(1+s)Y'(s) + (2+3s)Y(s) = 6.$$

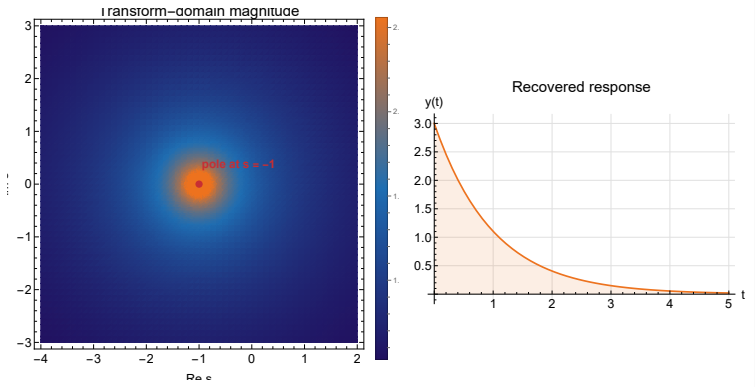
The original second-order time-domain equation has become a first-order linear equation for $Y(s)$.

Solving the s-domain equation gives a one-parameter family. The condition $y(t) \to 0$ and the final-value relation remove the inadmissible term, yielding

$$Y(s) = \frac{3}{s+1}, \qquad \boxed{y(t) = 3\mathrm{e}^{-t}}.$$

Direct substitution verifies both the ODE and $y(0) = 3$.

The transform singularity predicts the recovered decay



The simple pole at $s = -1$ corresponds to the exponential mode $3\mathrm{e}^{-t}$.

Why the extension stops here

Polynomial-coefficient equations can lead to Bessel functions, branch points, and nonrational transforms. These are valuable advanced topics, but they are not required for the main engineering initial-value problem workflow.

- retain the multiplication theorem as a useful extension;
- recognize when the transformed equation is differential;
- leave systematic special-function theory to advanced study and L16 review.

Which route should we choose?

Problem feature	Recommended route
standard constant-coefficient initial-value problem	direct transform and partial fractions
many inputs for one model	Green function / impulse response
steps or pulses	time-shift representation
impulsive input	transform the delta directly
coupled equations	matrix algebra in s
polynomial coefficients	multiplication theorem, then solve for Y

Common errors and how to prevent them

- **Missing initial terms:** write the derivative-transform formula first.
- **Wrong local time:** after e^{-as}, use $t - a$ inside the delayed response.
- **Unfactored denominator:** identify pole multiplicities before inversion.
- **Lost causality:** include the unit step in every delayed term.
- **Unchecked result:** verify the ODE, initial data, and pole-implied behavior.

Complete ODE solution workflow

1. formulate the model and record all initial data;
2. rewrite switching, pulses, and impulses in transform-ready form;
3. transform term by term and solve for the unknown transform;
4. simplify using poles, partial fractions, shifts, or matrix algebra;
5. invert and restore causal factors;
6. verify quantitatively and interpret physically.

Summary

- Constant-coefficient IVPs become algebraic equations containing the initial data.
- Poles organize natural modes, damping, resonance, and stability behavior.
- Steps and impulses model switching and instantaneous inputs.
- Systems of ODEs become matrix equations in the transform domain.
- Mechanical and electrical models share one mathematical structure.
- Polynomial coefficients provide a useful but limited extension.

Next $\gg$ Lecture 14 — Systems, Convolutions, and PDEs