L14 — Systems, Convolutions, and PDEs

Engineering Mathematics I

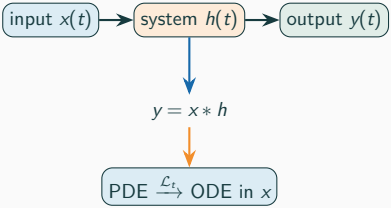
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Road map

- LTI systems and transfer functions
- Convolution in system analysis
- Partial differential equations



From solving one ODE to describing a system

L13 solved individual ODE models. Here we ask a broader question:

How does one model respond to every admissible input?

For an Linear time invariant (LTI) system, the answer is encoded by one function, the impulse response $h(t)$:

$$y(t) = (h * x)(t).$$

The same transform strategy will then be extended from system models to convolution equations and PDEs.

Teaching progression

Representation $\rightarrow$ response calculation $\rightarrow$ distributed-parameter models.

Zero-state and zero-input responses

For a linear model, separate the response caused by stored energy from the response caused by the applied input:

$$y(t) = y_{\mathrm{IC}}(t) + y_{\mathrm{ZS}}(t), \qquad Y(s) = Y_{\mathrm{IC}}(s) + H(s)X(s).$$

Zero-input response

$x(t) = 0 \implies y_{\mathrm{IC}}(t)$ from the initial state

Natural modes carry the stored energy.

Zero-state response

$$y_{\mathrm{ZS}}(t) = (h * x)(t), \qquad Y_{\mathrm{ZS}} = HX.$$

The transfer function describes this part.

LTI Systems and Transfer Functions

Linearity and time invariance

A system $\mathcal{S}$ is linear if

$$\mathcal{S}\{ax_1 + bx_2\} = a\mathcal{S}\{x_1\} + b\mathcal{S}\{x_2\},$$

and time invariant if

$$\mathcal{S}\{x(t - t_0)\} = y(t - t_0)$$

whenever $\mathcal{S}\{x(t)\} = y(t)$.

Engineering meaning

Superposition is valid, and the system behavior does not depend on the absolute starting time.

A system with both properties is called an LTI system.

The impulse response characterizes a causal LTI system

The impulse response $h(t)$ is the zero-state output of the system when the input is $\delta(t)$ (unit impulse), with the system initially at rest,

$$\mathcal{S}\{\delta(t)\} = h(t).$$

For causal input $x(t)$, the sifting property gives

$$x(t) = \int_0^t x(\tau)\delta(t-\tau)\,\mathrm{d}\tau.$$

By linearity and time invariance,

$$y(t) = \mathcal{S}\{x(t)\} = \mathcal{S}\left\{\int_0^t x(\tau)\delta(t-\tau)\,\mathrm{d}\tau\right\} = \int_0^t x(\tau)\mathcal{S}\{\delta(t-\tau)\}\,\mathrm{d}\tau,$$

i.e.,

$$y(t) = \int_0^t x(\tau)h(t-\tau)\,\mathrm{d}\tau.$$

Therefore

$$\boxed{y(t) = (h * x)(t)}.$$

The transfer function is the transform of the impulse response

For a causal LTI system with zero initial conditions,

$$y(t) = (h * x)(t).$$

Taking the Laplace transform, convolution becomes multiplication:

$$Y(s) = H(s)X(s).$$

The transfer function $H(s)$ is

$$\boxed{H(s) = \frac{Y(s)}{X(s)} = \mathcal{L}\{h(t)\}}.$$

So $h(t)$ describes the system in the time domain, and $H(s)$ describes it in the s-domain.

Important convention

$H(s)$ describes **zero-state** input–output behavior. Initial-condition terms belong to the complete response, but not to $H(s)$.

From a differential equation to a transfer function

For

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = b_m x^{(m)} + \cdots + b_0 x,$$

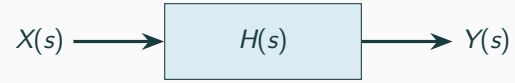
zero initial data give

$$a_n s^n Y(s) + \cdots + a_1 s Y(s) + a_0 Y(s) = b_m s^m X(s) + \cdots + b_1 s X(s) + b_0 X(s).$$

Thus, the transfer fuction $H(s)$ is

$$\boxed{H(s) = \frac{Y(s)}{X(s)} = \frac{b_m s^m + \cdots + b_1 s + b_0}{a_n s^n + \cdots + a_1 s + a_0}}.$$

Denominator roots are poles; numerator roots are zeros. Poles describe natural modes, while zeros shape how inputs reach the output.



$$Y(s) = H(s)X(s)$$

Worked example: a first-order low-pass model

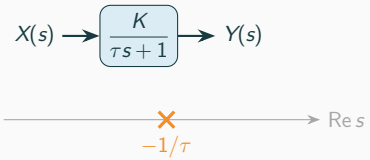
Consider the zero-state model

$$\tau y'(t) + y(t) = Kx(t), \qquad y(0) = 0, \qquad \tau > 0.$$

$$(\tau s + 1)Y(s) = KX(s),$$

$$H(s) = \frac{Y}{X} = \frac{K}{\tau s + 1},$$

$$h(t) = \frac{K}{\tau} \mathrm{e}^{-t/\tau} u(t).$$



Read the model before applying an input

The pole $-1/\tau$ sets the response speed; the DC gain $H(0) = K$ sets the steady amplification.

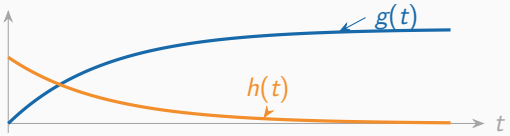
Step response and impulse response contain the same model

For the low-pass model and a unit-step input,

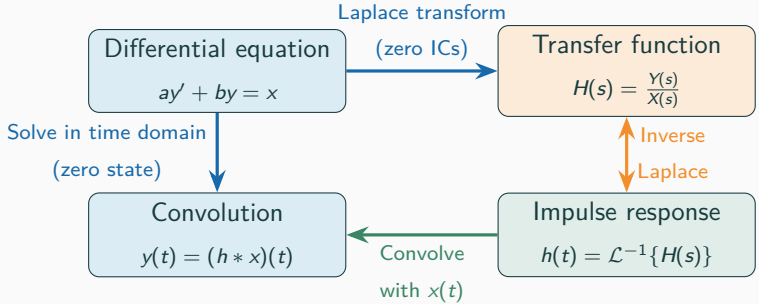
$$G(s) = \frac{K}{s(\tau s + 1)}, \qquad g(t) = K\bigl(1 - \mathrm{e}^{-t/\tau}\bigr)\,u(t).$$

More generally, a causal step response and impulse response satisfy

$$g(t) = \int_0^t h(\tau)\,\mathrm{d}\tau, \qquad h(t) = g'(t) \quad \text{away from jumps.}$$



Four equivalent views of the same zero-state LTI system



How to choose the right view

- **Given ODE:** use Laplace to get $H(s)$, or solve directly in time.
- **Given $H(s)$:** compute $Y(s) = H(s)X(s)$, then inverse transform.
- **Given $h(t)$:** convolve with the input: $y = h * x$.

Convolution in System Analysis

The unilateral convolution

For causal functions,

$$\boxed{(f * g)(t) = \int_0^t f(\tau) g(t-\tau)\,\mathrm{d}\tau}.$$

Its transform is

$$\mathcal{L}\{f * g\} = F(s)\,G(s).$$

L11 established the theorem; L14 uses it to compute system outputs and solve integral equations.

A finite-duration input makes the overlap visible

A causal rectangular pulse can be written as

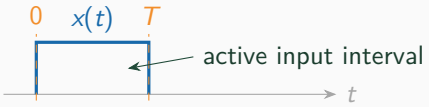
$$x(t) = u(t) - u(t - T).$$

For an impulse response $h(t)$,

$$y(t) = \int_0^t h(t - \tau)\,[u(\tau) - u(\tau - T)]\,\mathrm{d}\tau.$$

Hence

$$y(t) = \begin{cases} \displaystyle\int_0^t h(t - \tau)\,\mathrm{d}\tau, & 0 \le t < T, \\ \displaystyle\int_0^T h(t - \tau)\,\mathrm{d}\tau, & t \ge T. \end{cases}$$



Convolution geometry I: flip and shift

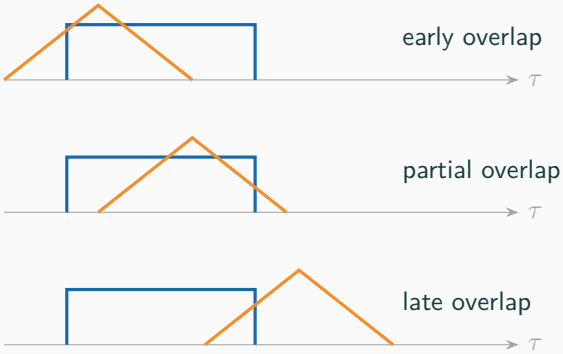
For fixed t, regard the convolution integrand as a function of τ:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,\mathrm{d}\tau.$$



Shifting the flipped response by t produces $h(t-\tau)$; it moves to the right as the observation time increases.

Convolution geometry II: overlap changes with time

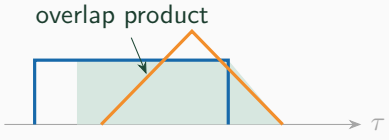


Reading rule

Multiply only where the two graphs overlap; the signed overlap area is the single output value $y(t)$.

Convolution geometry III: integrate the product

$$y(t) = \int x(\tau)h(t-\tau)\,\mathrm{d}\tau$$



1. flip $h(\tau)$ to obtain $h(-\tau)$;
2. shift to $h(t-\tau)$;
3. multiply by $x(\tau)$;
4. integrate over the overlap.

Repeating the four steps for every t builds the complete output curve.

Algebraic properties simplify system calculations

For suitable causal functions,

$$
\begin{aligned}
f * g &= g * f, \\
(f * g) * h &= f * (g * h), \\
f * (g + h) &= f * g + f * h, \\
f * \delta &= f.
\end{aligned}
$$

These correspond to multiplication properties in the transform domain. Associativity explains why cascaded LTI systems may be grouped without changing the overall impulse response.

Cascaded systems convolve their impulse responses



$$h_{\text{eq}} = h_1 * h_2$$

In the s-domain,

$$H_{\text{eq}}(s) = H_1(s)H_2(s).$$

For example,

$$H_1(s) = \frac{1}{s+1}, \quad H_2(s) = \frac{1}{s+2}$$

$$\implies \quad H_{\text{eq}}(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}.$$

Inverse transform:

$$h_{\text{eq}}(t) = \left(e^{-t} - e^{-2t}\right)u(t).$$

Worked example: exponential input through a first-order system

Let

$$h(t) = \mathrm{e}^{-2t}u(t), \qquad x(t) = \mathrm{e}^{-t}u(t).$$

Then

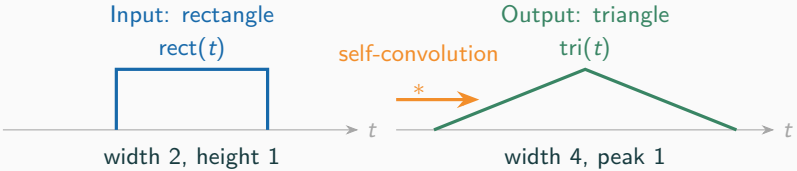
$$y(t) = \int_0^t \mathrm{e}^{-2(t-\tau)}\mathrm{e}^{-\tau}\,\mathrm{d}\tau = \mathrm{e}^{-t} - \mathrm{e}^{-2t}.$$

Equivalently,

$$H(s)X(s) = \frac{1}{(s+2)(s+1)}.$$

The two methods expose the same pair of decaying modes.

Convolution smooths sharp inputs



$$(\text{rect} * \text{rect})(t) = \int_{-\infty}^{\infty} \text{rect}(\tau)\,\text{rect}(t-\tau)\,d\tau = \text{tri}(t)$$

Intuition: overlap area

Convolution = flip, shift, multiply, integrate.

- Perfect overlap: the area is largest.
- Shift left or right: the overlap decreases linearly.
- Output rises, then falls $\Rightarrow$ a triangle.

Key pattern

Each additional convolution gives:

more smoothness + wider support

rect $\rightarrow$ tri $\rightarrow$ smoother splines $\rightarrow \cdots$

Step response, impulse response, or direct convolution?

Available information	Efficient route
rational $H(s)$ and simple $X(s)$	multiply and invert
explicit $h(t)$ and finite-support $x(t)$	integrate overlap directly
measured step response $g(t)$	differentiate to obtain $h(t)$
cascade of subsystems	multiply H_k or convolve h_k

Convolution is powerful but its conditions matter

Check that:

- the signals are causal or the correct bilateral definition is used;
- the integral exists on the interval of interest;
- initial-condition contributions are not hidden inside a transfer function;
- delayed inputs use the correct local time;
- units remain consistent: h may carry inverse-time units.

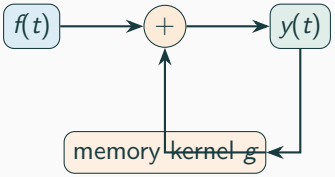
Convolution application: a Volterra equation

A causal memory equation has the form

$$y(t) = f(t) + \int_0^t g(t-\tau)y(\tau)\,\mathrm{d}\tau = f(t) + (g * y)(t).$$

Transforming turns the history integral into multiplication:

$$Y = F + GY, \qquad \boxed{Y(s) = \frac{F(s)}{1 - G(s)}}.$$



Worked example: transform the memory equation

Solve

$$y(t) = t^2 + \int_0^t y(\tau)\cos(t-\tau)\,\mathrm{d}\tau.$$

With $F(s) = 2/s^3$ and $G(s) = s/(s^2+1)$,

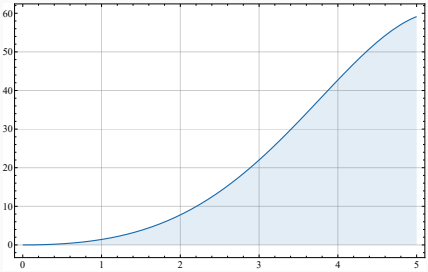
$$Y = \frac{2}{s^3} + \frac{s}{s^2+1}Y, \qquad Y(s) = \frac{2(s^2+1)}{s^3(s^2-s+1)}.$$

Partial fractions and completion of the square then prepare the expression for table-based inversion.

Worked example: solution and verification

$$y(t) = 2 + 2t + t^2 - 2e^{t/2}\left[\cos\frac{\sqrt{3}t}{2} + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}t}{2}\right].$$

- $y(0) = 0$, consistent with the original equation;
- transforming the result recovers $Y = F + GY$;
- the kernel feeds the past response back into the present value.



Partial differential equations

A transform in time reduces a PDE to an ODE in space

Let

$$U(x,s) = \mathcal{L}_t\{u(x,t)\}.$$

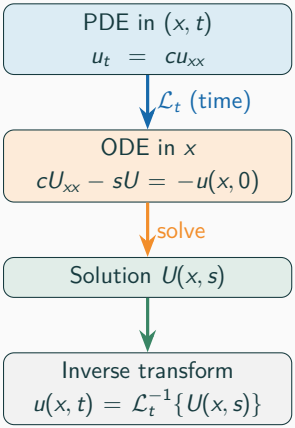
The time Laplace transform gives

$$\mathcal{L}_t\{u_t\} = sU - u(x,0), \qquad \mathcal{L}_t\{u_{xx}\} = U_{xx}.$$

- Time derivative $\to$ algebraic in s.
- Space derivative $\to$ still a derivative in x.

Applying this to the heat equation

$$u_t = cu_{xx} \quad \Longrightarrow \quad sU - u(x,0) = cU_{xx}.$$



Key idea

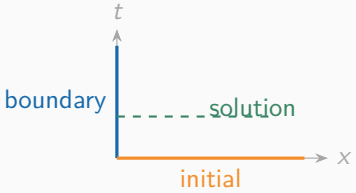
Laplace transform in t converts time derivatives into algebraic terms, so a PDE becomes an ODE in the remaining spatial variable x.

Initial and boundary data play different roles

- Initial data enter through transformed time derivatives.
- Boundary data select the spatial solution.
- Boundedness removes growing spatial modes.
- Invert only after the spatial problem is complete.

Key point

The PDE alone does not determine a unique physical solution.



Two fundamental evolution equations

The one-dimensional heat equation is

$$u_t = cu_{xx}, \qquad c > 0,$$

while the wave equation is

$$y_{tt} = a^2 y_{xx}, \qquad a > 0.$$

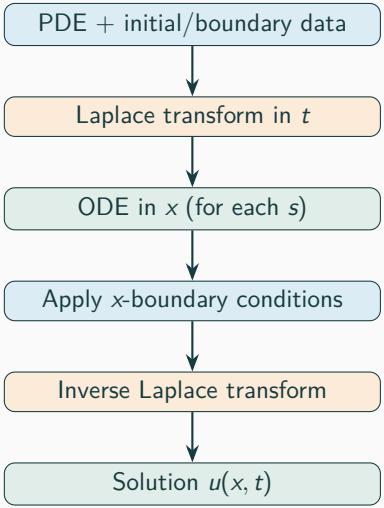
Heat

1st order in t $\Rightarrow$ smoothing

Wave

2nd order in t $\Rightarrow$ propagation

The PDE transform workflow



Why this works

- **Transform in** t: turns u_t into $sU - u(x,0)$.
- **Result**: time derivatives disappear; the PDE becomes an ODE in x.
- **Boundaries**: solve the ODE and use the given x-conditions to find constants.
- **Invert in** s: return to the original time variable.
- **Verify**: substitute back into the PDE and check the data.

Key idea

Laplace transform in t converts a PDE into an ODE in x.

Heat equation on a semi-infinite rod

Consider

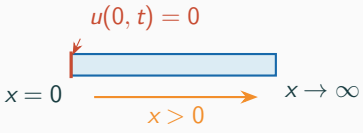
$$u_t = cu_{xx}, \qquad x > 0, \;\; t > 0,$$

with

$$u(x,0) = 1, \qquad u(0,t) = 0, \qquad u(x,t) \text{ bounded as } x \to \infty.$$

Physically, an initially uniform rod has its end at $x = 0$ suddenly maintained at zero temperature.

$$\underbrace{u(x,0) = 1}_{\text{initial state}}, \qquad \underbrace{u(0,t) = 0}_{\text{cooled boundary}} .$$



Heat equation: Laplace transform and spatial conditions

Transformation in t gives

$$sU(x,s) - 1 = cU_{xx}(x,s), \qquad U_{xx} - \frac{s}{c}U = -\frac{1}{c}.$$

A particular solution is $U_p = 1/s$. The homogeneous spatial modes are $\exp(\pm x\sqrt{s/c})$, so

$$U(x,s) = \frac{1}{s} + A(s)\mathrm{e}^{x\sqrt{s/c}} + B(s)\mathrm{e}^{-x\sqrt{s/c}}.$$

The integration constants may depend on s because only the spatial variable has been integrated. For $\operatorname{Re} s > 0$, the factor $\mathrm{e}^{x\sqrt{s/c}}$ grows as $x \to \infty$. Boundedness therefore requires $A(s) = 0$. The cooled-end condition becomes $U(0,s) = 0$, hence

$$0 = \frac{1}{s} + B(s), \qquad B(s) = -\frac{1}{s}.$$

The transformed boundary-value problem is now complete:

$$\boxed{U(x,s) = \frac{1}{s}\left(1 - \mathrm{e}^{-x\sqrt{s/c}}\right).}$$

Heat equation: invert, verify, and interpret

Using the standard inverse pair involving the error function,

$$\boxed{u(x,t) = \operatorname{erf}\left(\frac{x}{2\sqrt{ct}}\right)}$$

where

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-s^2}\, ds.$$

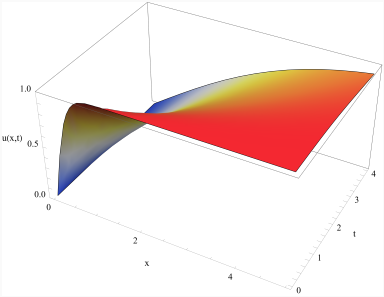
Mathematical checks

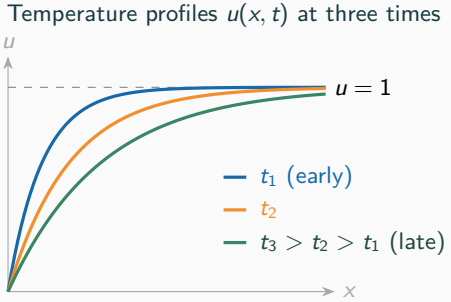
$u(0,t) = 0$, $\lim_{x\to\infty} u(x,t) = 1$, and for fixed $x > 0$, $u(x,t) \to 1$ as $t \downarrow 0$.

Physical reading

The transition layer has thickness $O(\sqrt{ct})$; diffusion has no sharply defined propagation front.

Heat diffusion: no finite-speed front





Spacetime surface $u(x,t)$

The boundary influence spreads into larger x; there is no finite front.

Fixed-time snapshots

At later times the same temperature change reaches farther into the rod; the profile broadens without translating as a rigid front.

Diffusion has a characteristic length scale

The similarity variable is

$$\eta = \frac{x}{2\sqrt{ct}}.$$

Therefore profiles at different times collapse when plotted against $x/\sqrt{t}$.

- larger diffusivity spreads temperature changes faster;
- the affected length is proportional to $\sqrt{ct}$;
- high spatial frequencies decay rapidly.

A fixed string: one exact vibration mode

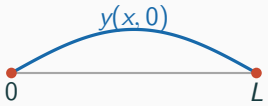
Consider

$$y_{tt} = a^2 y_{xx}, \qquad 0 < x < L,$$

with fixed ends and initial data

$$y(0,t) = y(L,t) = 0, \qquad y(x,0) = \sin\frac{\pi x}{L}, \qquad y_t(x,0) = 0.$$

This isolates the first spatial mode and keeps the transform calculation transparent. The endpoints are nodes for all t.



$$\text{spatial shape } \sin\frac{\pi x}{L} \quad \times \quad \text{time factor } \cos\frac{a\pi t}{L}.$$

Wave equation: transform and solve

Since $y(x,0) = \sin(\pi x/L)$ and $y_t(x,0) = 0$, transformation in time gives

$$s^2 Y - s \sin \frac{\pi x}{L} = a^2 Y_{xx}$$

$$Y_{xx} - \frac{s^2}{a^2} Y = -\frac{s}{a^2} \sin \frac{\pi x}{L}.$$

Because the forcing is one spatial eigenmode, seek $Y(x,s) = C(s)\sin(\pi x/L)$. Substitution gives

$$\left[s^2 + \left(\frac{a\pi}{L} \right)^2 \right] C(s) = s.$$

Wave equation: boundary conditions and inversion

The sine mode already satisfies $Y(0,s) = Y(L,s) = 0$, so

$$Y(x,s) = \frac{s}{s^2 + (a\pi/L)^2} \sin \frac{\pi x}{L}.$$

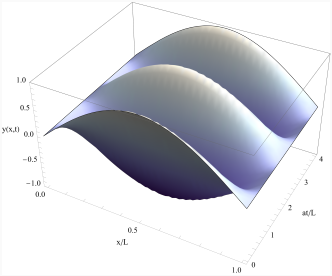
Inverting the remaining scalar transform gives

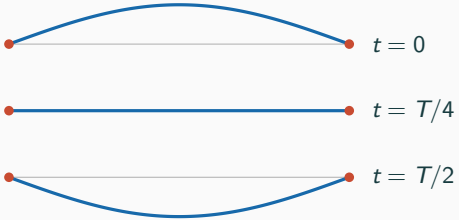
$$\boxed{y(x,t) = \sin \frac{\pi x}{L} \cos \frac{a\pi t}{L}}.$$

Interpretation

The spatial eigenfunction is preserved; only its amplitude oscillates. The fixed ends remain nodes and the ideal model has no damping.

Wave modes: fixed nodes, oscillating amplitude





Spacetime surface

Nodes remain fixed while the amplitude oscillates; ideal waves do not dissipate energy.

Static snapshots

The nodes stay fixed while the displacement changes sign. This is oscillatory propagation, not diffusive smoothing.

Heat and wave equations encode different physics

Feature	Heat equation	Wave equation
time derivative	first order	second order
initial data	one field	displacement and velocity
behavior	smoothing and decay	propagation and oscillation
natural scale	$x/\sqrt{t}$	$x \pm at$ or modal periods

Boundary conditions represent physical devices

For heat flow:

- prescribed temperature: $u(0,t) = T_0$;
- insulated end: $u_x(0,t) = 0$;
- convective boundary: $-ku_x = h(u - T_\infty)$.

For strings:

- fixed end: $y = 0$;
- free end: $y_x = 0$.

The same PDE can produce qualitatively different solutions under different devices.

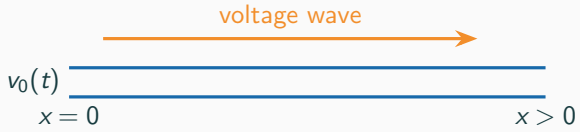
Engineering bridge: a lossless transmission line

With inductance ℓ and capacitance c_0 per unit length,

$$v_x = -\ell i_t, \qquad i_x = -c_0 v_t.$$

Eliminating the current gives a wave equation for voltage:

$$v_{xx} = \ell c_0 v_{tt}, \qquad c_p = \frac{1}{\sqrt{\ell c_0}}.$$

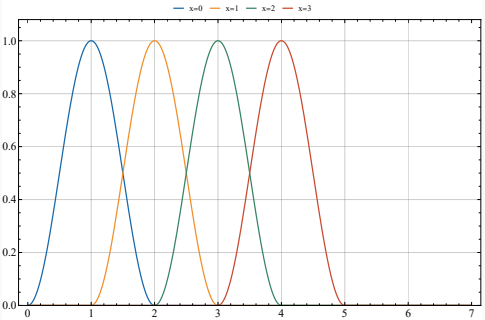


The input waveform propagates with a delay

For a line initially at rest and a bounded right-traveling solution,

$$v(x,t) = v_0\left(t - \frac{x}{c_p}\right) u\left(t - \frac{x}{c_p}\right).$$

- the waveform shape is preserved in the ideal model;
- distance appears as a propagation delay;
- losses and reflections require a richer model.



Method-selection guide

Question	Primary representation	Required check
output of an LTI model	$Y = HX$	zero-state convention
time-domain overlap	$y = h * x$	causal integration limits
causal memory equation	$Y = F + GY$	solve for Y before inversion
PDE evolution problem	ODE in space	all initial/boundary data

One recurring principle

Transform first only when the transformed problem is simpler and all data can be carried into the new domain without loss.

Common errors across systems and PDEs

- defining $H(s)$ with nonzero initial conditions;
- using bilateral convolution limits for a unilateral causal problem;
- forgetting that integration constants in a spatial ODE may depend on s;
- applying only the initial data but not the boundary conditions;
- accepting a spatially growing exponential in a bounded domain problem;
- treating a truncated Fourier series as an exact jump.

Summary

- A causal LTI system is described by $h(t)$ or $H(s)$.
- Convolution converts an input and impulse response into the output.
- A convolution-type Volterra equation becomes algebraic in s.
- Transforming a PDE in t produces an ODE in x.
- Initial, boundary, boundedness, and decay conditions are all essential.
- Heat and wave equations illustrate diffusion versus oscillatory propagation.
- A lossless transmission line provides an engineering wave-propagation bridge.

Next $\gg$ Lecture 15 — Control Systems and Engineering Applications