L16 — Advanced Topics and Review

Engineering Mathematics I

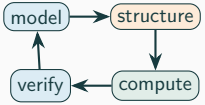
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Road map

- Advanced Complex Analysis Topics
- Asymptotic Analysis
- Numerical Inverse Laplace Transform
- Summary and Review
- Future Directions



Use and check

Cartesian form is best for components and distance; polar form is best for products, quotients, powers and roots. Remember $1/z = \bar{z}/|z|^2$ only for $z \neq 0$.

The course developed two complementary languages

Complex-variable language

analyticity $\Rightarrow$ integrals

$\Rightarrow$ series and residues

It exposes geometry, singularities and global integral information.

Transform language

model $\Rightarrow$ frequency algebra

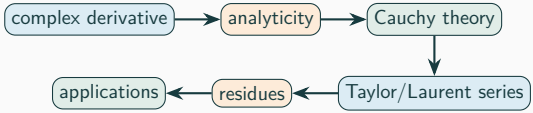
$\Rightarrow$ inverse and interpret

It exposes modes, spectra, responses and stability.



Advanced Complex Analysis Topics

Complex analysis is a local-to-global theory



Unifying point

Local differentiability constrains a function strongly enough to determine contour integrals, series coefficients and many global properties.

Choose the theorem from the analytic domain

Structure detected	Preferred tool	Required check
analytic inside C	Cauchy theorem/formula	domain, orientation
isolated poles	residue theorem	enclosed singularities
local behavior	Taylor/Laurent series	center, annulus
zero counting	argument principle	boundary zeros/poles
angle-preserving map	conformal mapping	$f'(z) \neq 0$

Analytic continuation extends a local formula

A power series centered at z_0 initially defines

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \qquad |z - z_0| < R_0.$$

If a series centered at z_1 agrees on an overlapping open set, both formulas represent the same analytic function there.

Continuation principle

Move through overlapping disks while uniqueness preserves the function. A regular point on the first disk boundary can be crossed. Singularities and path dependence constrain further continuation.

Uniqueness makes continuation reliable

If analytic functions f and g agree on a set with an accumulation point inside connected D, then $f = g$ throughout D.

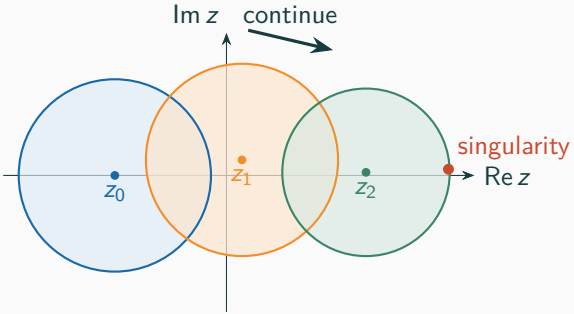
Guarantee

Compatible local formulas cannot generate two different continuations on the same connected region.

Obstruction

Poles, branch points and incompatible paths around multivalued functions require additional domain choices.

Overlapping disks carry information across the plane



1. expand locally;
2. enter an overlap;
3. re-expand;
4. stop at a singularity.

Worked example: a disk boundary need not be a function boundary

Around $z = 0$,

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \qquad |z| < 1.$$

Re-expanding around $z = -1$ gives

$$\frac{1}{1-z} = \frac{1}{2-(z+1)} = \frac{1}{2}\sum_{n=0}^{\infty} \left(\frac{z+1}{2}\right)^n, \qquad |z+1| < 2.$$

The disks overlap, so the second series continues the first beyond $|z| = 1$. The actual obstruction is the pole at $z = 1$.

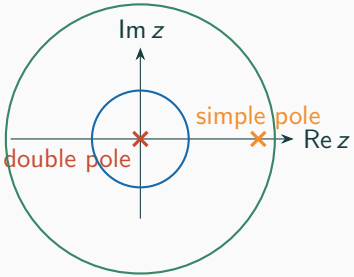
Earlier applications now occupy one method map

Application	Governing structure	Primary lecture
real definite integrals	residues and arc estimates	L08
transcendental root counts	argument principle	L08
Gamma-function identities	continuation and residues	L08
potential flow	analytic potentials and maps	L07–L08
local approximation	Taylor/Laurent expansions	L05–L06

L16 connects these ideas without repeating their detailed calculations.

Checkpoint: diagnose before calculating

$$f(z) = \frac{e^z}{z^2(z-2)}$$



Question	Tool selected
near $z = 0$	Laurent/local residue
on $\lvert z\rvert = 1$	residue at 0
on $\lvert z\rvert = 3$	residues at 0 and 2
count zeros/poles	argument principle

The expression is fixed; the question and domain select the method.

Asymptotic Analysis

Exact formulas are not always the useful endpoint

$$t \to 0^+, \qquad t \to \infty, \qquad |s| \to \infty, \qquad \varepsilon \to 0^+.$$



$$f(x) = f_0(x) + f_1(x) + R_1(x), \qquad f_1 = o(f_0).$$

A useful approximation must say

which limit is taken, where it is valid, and how large the neglected remainder is.

Asymptotic equality is an ordered approximation

We write

$$f(x) \sim \sum_{n=0}^{\infty} a_n \phi_n(x), \qquad x \to x_0,$$

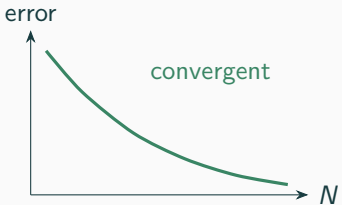
when $\phi_{n+1} = o(\phi_n)$ and, for every fixed N,

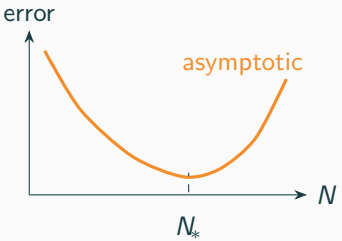
$$f(x) - \sum_{n=0}^{N} a_n \phi_n(x) = o(\phi_N(x)).$$

Distinction

The definition tests each fixed truncation in a limit; it does not require the infinite series to converge.

Convergent and asymptotic series answer different questions



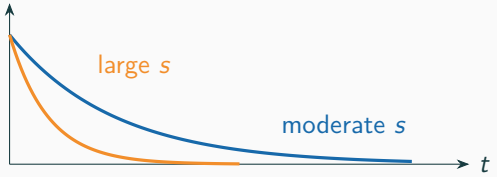


More terms always help a convergent expansion inside its domain; an asymptotic expansion is often best near its smallest term.

Large transform frequency probes short time

The factor e^{-st} localizes $F(s) = \int_0^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t$ near $t = 0$ when s is large:

$$|s| \to \infty \quad \Longleftrightarrow \quad t \to 0^+.$$



Endpoint coefficients give large-s coefficients

Suppose f is locally integrable, of exponential order at infinity, and

$$f(t) = \sum_{n=0}^{N} a_n t^{\nu+n} + O(t^{\nu+N+1}), \quad t \to 0^+, \quad \nu > -1.$$

Then, along the positive real axis,

$$F(s) = \sum_{n=0}^{N} \frac{a_n \Gamma(\nu+n+1)}{s^{\nu+n+1}} + O(s^{-\nu-N-2}), \quad s \to +\infty.$$

Reason

Integrate each endpoint power against e^{-st}. The part away from the endpoint is exponentially small for large positive s.

Worked example: prepare a large-s expansion

$$F(s) = \frac{1}{\sqrt{s^2+1}} = \frac{1}{s}\left(1+\frac{1}{s^2}\right)^{-1/2}.$$

Using $(1+x)^{-1/2} = 1 - x/2 + 3x^2/8 - 5x^3/16 + \cdots$ gives

$$F(s) \sim \frac{1}{s} - \frac{1}{2s^3} + \frac{3}{8s^5} - \frac{5}{16s^7} + \cdots.$$

The inverse powers of s are now ready for termwise inversion.

Worked example: invert term by term and verify

Since $\mathcal{L}^{-1}\{s^{-(n+1)}\} = t^n/n!$,

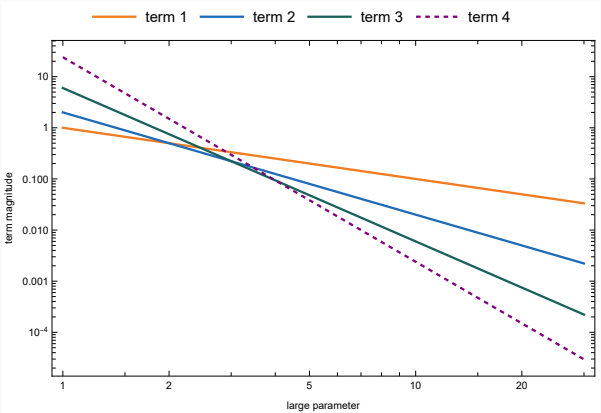
$$f(t) \sim 1 - \frac{t^2}{4} + \frac{t^4}{64} - \frac{t^6}{2304} + \cdots .$$

This is the short-time expansion of $J_0(t)$.

Checks

$f(0) = 1$; only even powers appear; and forward transformation reproduces the large-s expansion of $F(s)$.

Successive terms reveal a useful stopping point



1. fix the parameter;
2. compare term magnitudes;
3. stop near the smallest term;
4. estimate error by the next term.

An asymptotic result must carry its validity statement

Report $f(x) = f_N(x) + R_N(x)$ with a stated limit and

$$R_N(x) = O(\phi_{N+1}(x)) \,.$$

Question	Required answer
What is small or large?	limiting parameter
Where is it valid?	domain, sector or time interval
How many terms are useful?	truncation rule
How is it checked?	exact, numerical or governing equation

Numerical Inverse Laplace Transform

Numerical inversion answers a different question

An asymptotic expansion describes a limit. A numerical inverse estimates $f(t)$ at specified positive times from values of $F(s)$.

$$f(t) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} \mathrm{e}^{st} F(s)\, \mathrm{d}s.$$

Connection to earlier methods

Cauchy theory permits a contour deformation when singularities are respected and the connecting tails vanish. Quadrature then replaces the integral by samples.

Task

Choose a valid contour and test accuracy over the requested time interval.

Numerical methods make different compromises

Family	Main idea	Typical caution
real-axis sampling	weighted samples of F	cancellation
basis expansion	approximate f in a basis	parameter selection
Fourier-series method	discretize a vertical contour	aliasing
deformed contour	move into a decaying region	singularity geometry

The Talbot family connects numerical computation to the contour geometry of L12.

Talbot inversion: contour conditions come first

For $t > 0$, one representative contour is

$$s(\theta) = \mu(\theta \cot \theta + \mathrm{i}\theta), \quad -\pi < \theta < \pi, \quad \mu > 0.$$

1. Locate all poles and branch cuts of F.
2. Check that the deformation preserves the Bromwich integral.
3. Check decay of $\mathrm{e}^{st}F(s)$ on the tails.

Condition

The formula for $s(\theta)$ alone does not guarantee a valid inverse. Contour scale and singularity geometry must be compatible at each time.

Talbot inversion: the contour becomes a finite sum

After a valid deformation,

$$f(t) = \frac{1}{2\pi \mathrm{i}} \int_{-\pi}^{\pi} \mathrm{e}^{s(\theta)t} F(s(\theta)) s'(\theta)\, \mathrm{d}\theta.$$

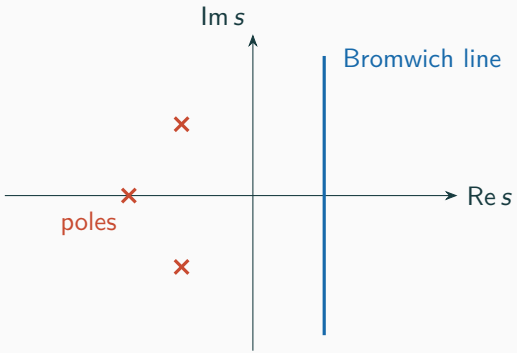
Choose midpoint nodes $\theta_k = -\pi + (k - \frac{1}{2})\Delta\theta$, where $\Delta\theta = 2\pi/N$ and $k = 1, \ldots, N$. Then

$$f_N(t) = \frac{\Delta\theta}{2\pi \mathrm{i}} \sum_{k=1}^{N} \mathrm{e}^{s(\theta_k)t} F(s(\theta_k)) s'(\theta_k).$$

Implementation check

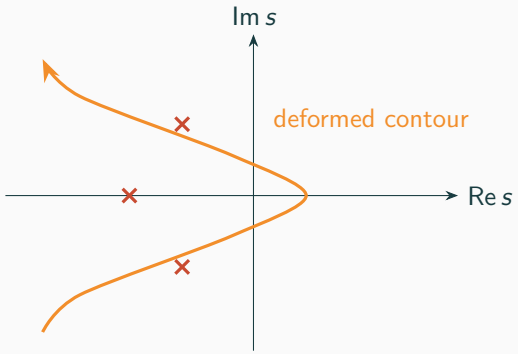
Avoid the endpoints. At $\theta = 0$, use the removable limits $s(0) = \mu$ and $s'(0) = \mathrm{i}\mu$. Increase working precision when cancellation becomes significant.

Talbot key frame 1: inspect singularities first



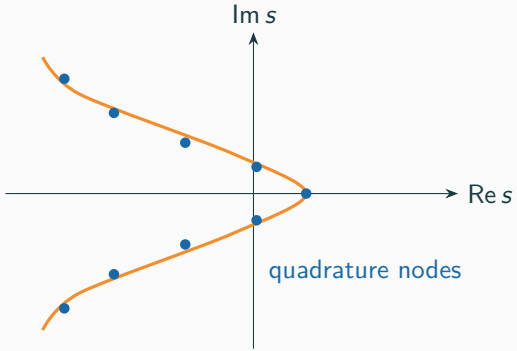
Place γ to the right of every singularity before deformation. Crossing a pole without its residue changes the inverse.

Talbot key frame 2: deform without crossing singularities



The center passes to the right of the singularities; the tails enter the left half-plane where e^{st} decays.

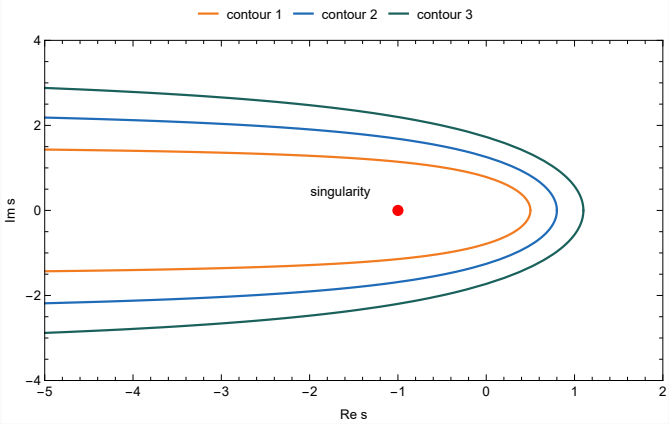
Talbot key frame 3: replace the contour by samples



$$f_N(t) = \sum_{k=1}^{N} w_k(t) F(s_k).$$

Increase N and precision together, then test sensitivity to μ.

Contour geometry confirms the deformation idea



Read the figure

Locate poles and branch cuts first. A useful contour approaches important singularities without crossing them and damps the exponential tails.

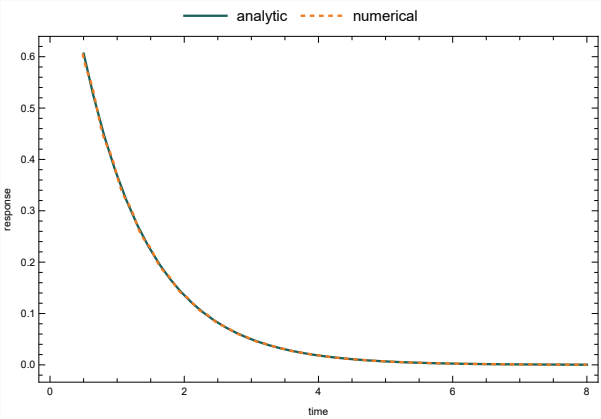
Worked numerical example: invert a known transform

Use $F(s) = 1/(s+1)$, whose exact inverse is $f(t) = \mathrm{e}^{-t}$.

1. identify the pole at $s = -1$;
2. choose a Talbot contour with the correct enclosure;
3. evaluate $F(s_k)$ at the quadrature nodes;
4. form $f_N(t) = \sum w_k F(s_k)$;
5. compare with e^{-t} while varying N, μ and precision.

A method that fails on a known inverse is not ready for an unknown one.

Accuracy must be checked over the required time interval



- compare pointwise error;
- test early and late time;
- vary precision;
- verify by the governing equation or forward transform.

Four error sources require four diagnostics

Error source	Symptom	Diagnostic
truncation	changes with N	increase nodes
roundoff	digits fluctuate	raise precision
poor contour	isolated large errors	inspect singularities
cancellation	late-time loss	rescale or compare methods

Agreement with an independent equation, limit or transform is stronger than agreement between two nearby parameter choices.

Summary and Review

Review block A: complex numbers, functions and analyticity

Geometry and algebra

$$z = x + \mathrm{i}y = r\mathrm{e}^{\mathrm{i}\theta},$$

$$|zw| = |z||w|, \qquad \arg(zw) = \arg z + \arg w.$$

Polar form turns multiplication, roots and powers into geometry.

Analyticity

For the function $f = u + \mathrm{i}v$ with continuous first partial derivatives,

$$u_x = v_y, \qquad u_y = -v_x$$

is the practical local test for complex differentiability.

Always state the domain: branches and excluded points are part of the function.

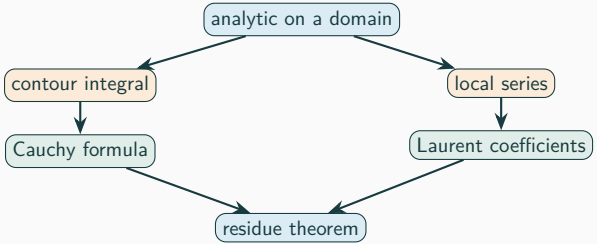
Elementary functions: identities survive, geometry changes

Function	Structural fact	Domain issue
e^z	entire, $2\pi\mathrm{i}$-periodic, never zero	non-injective
$\sin z, \cos z$	entire; exponential definitions	unbounded off real axis
$\operatorname{Log} z$	$(\operatorname{Log} z)' = 1/z$	branch cut required
z^α	$\exp(\alpha \operatorname{Log} z)$	branch inherited from Log

Fast check

Locate zeros and branch points before differentiating, integrating or composing.

Review block B: Cauchy theory, series and residues



$$\oint_C f(z)\,\mathrm{d}z = 2\pi\mathrm{i} \sum_{z_k \in \operatorname{int} C} \operatorname{Res}(f, z_k),$$

provided the contour avoids singularities and orientation is accounted for.

Series identify both approximation and singular structure

Taylor series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \qquad a_n = \frac{f^{(n)}(z_0)}{n!}$$

valid up to the nearest singularity.

Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$

$$a_{-1} = \operatorname{Res}(f, z_0),$$

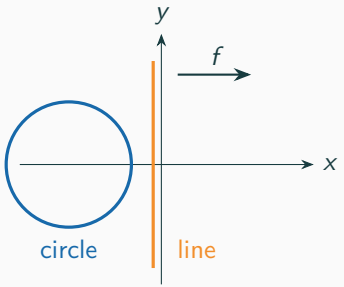
valid on a specified annulus.

Common error

A correct algebraic expansion with the wrong disk or annulus is an incomplete answer.

Review block C: mappings and complex-analysis applications

- $f'(z_0) \neq 0$ gives local angle preservation.
- Möbius maps send generalized circles to generalized circles.
- Boundary correspondence determines many canonical maps.
- Residues convert selected real integrals into contour integrals.
- The argument principle counts zeros minus poles.



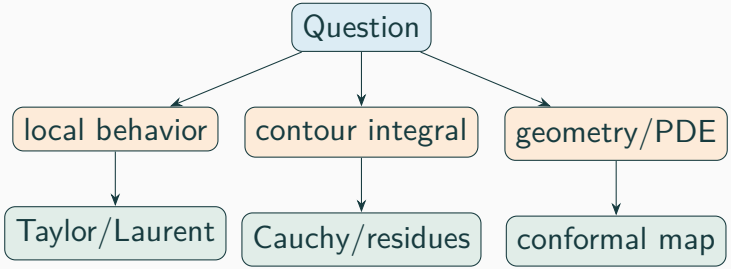
A contour-method checklist prevents most mistakes

1. Extend the real integrand to a complex function.
2. Choose a contour that matches decay, symmetry and branch structure.
3. Mark every enclosed pole and every excluded branch point.
4. Estimate auxiliary arcs before discarding them.
5. Compute residues with the correct pole order.
6. Recover the requested real quantity and check its sign and scale.

$$N_Z - N_P = \frac{1}{2\pi \mathrm{i}} \oint_C \frac{f'(z)}{f(z)}\,\mathrm{d}z$$

requires that f have no zeros or poles on C.

Complex-analysis method map



Domain, singularities and boundary geometry decide the method before algebra begins.

Review block D: Fourier analysis

Periodic signals

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \bigl(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\bigr).$$

Symmetry can eliminate half of the coefficient work.

Aperiodic signals

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} f(t) \mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t,$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(\omega) \mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega.$$

State the adopted normalization before applying a table or Parseval's identity.

Fourier methods connect waveform, spectrum and energy

Observation	Mathematical feature	Tool
periodic waveform	discrete harmonics	Fourier series
isolated pulse	continuous spectrum	Fourier transform
time shift	phase factor $\mathrm{e}^{-\mathrm{i}\omega t_0}$	shift property
smoothing/filtering	multiplication by $H(\mathrm{i}\omega)$	convolution theorem
energy comparison	L^2 norm equality	Parseval

At a jump, a Fourier series converges to the midpoint of the one-sided limits.

Review block E: Laplace foundations, properties and inversion

$$F(s) = \int_{0^-}^{\infty} \mathrm{e}^{-st} f(t)\,\mathrm{d}t, \qquad s = \sigma + \mathrm{i}\omega.$$

Forward

Check piecewise continuity, exponential order and ROC.

Operate

Use shifts, derivatives, integrals, periodicity and convolution.

Invert

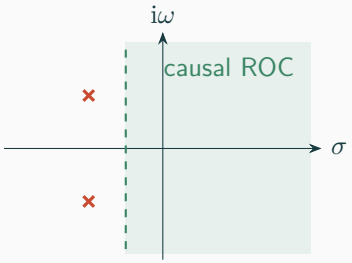
Simplify first; then use tables, partial fractions, convolution or residues.

ROC and pole geometry carry information beyond the formula

The same rational expression can represent different bilateral signals unless the ROC is stated. For causal engineering models, the ROC lies to the right of the rightmost pole.

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0^-),$$

$$\mathcal{L}\{f * g\} = F(s)G(s).$$



Fourier–Laplace method map

Fourier viewpoint

Use for periodic content, spectra, steady oscillations, filtering and energy distribution.

$$f(t) \longleftrightarrow \widehat{f}(\omega).$$

Laplace viewpoint

Use for causal transients, initial conditions, switching, impulses and modes.

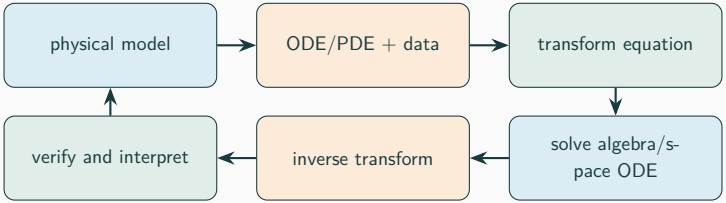
$$f(t) \longleftrightarrow F(s), \quad s = \sigma + \mathrm{i}\omega.$$

Fourier data can be a boundary slice of a Laplace transform when the ROC reaches the imaginary axis.

Transform properties form an operational vocabulary

Time-domain feature	Transform-domain feature	Main use
delay	$\mathrm{e}^{-as}F(s)$	switching
derivative	$sF(s) - f(0^+)$	IVPs
convolution	$F(s)G(s)$	systems/Green functions
exponential weight	$F(s-a)$	shifted poles
periodicity	line spectrum/formula	steady signals

Review block F: ODEs, PDEs, systems and control



$$Y(s) = H(s)X(s), \qquad T(s) = \frac{G(s)}{1 + G(s)H_f(s)}.$$

Initial/boundary conditions belong in the transformed model; poles then predict modes and stability.

Initial conditions and forcing contribute separately

$$y(t) = y_{\mathrm{IC}}(t) + y_{\mathrm{ZS}}(t), \qquad y_{\mathrm{ZS}} = h * x.$$

Structure	What it reveals	Essential check
poles	natural modes and stability	cancellations/ROC
zeros	blocked or reshaped inputs	physical realizability
convolution	accumulated impulse response	causality
feedback denominator	closed-loop modes	parameter range
PDE transform	spatial boundary-value problem	all boundary data

Worked example 1: test analyticity and recover a function

Let $u(x, y) = x^2 - y^2$. Determine whether it can be the real part of an analytic function.

$$u_x = 2x, \qquad u_y = -2y, \qquad u_{xx} + u_{yy} = 2 - 2 = 0.$$

Hence u is harmonic on $\mathbb{C}$. The Cauchy–Riemann equations require

$$v_y = u_x = 2x, \qquad v_x = -u_y = 2y.$$

Integrating v_y gives $v = 2xy + \phi(x)$.

Worked example 1: enforce compatibility and verify

Since $v_x = 2y + \phi'(x) = 2y$, we obtain $\phi' = 0$ and

$$v(x, y) = 2xy + C, \qquad f(z) = u + \mathrm{i}v = z^2 + \mathrm{i}C.$$

Checks

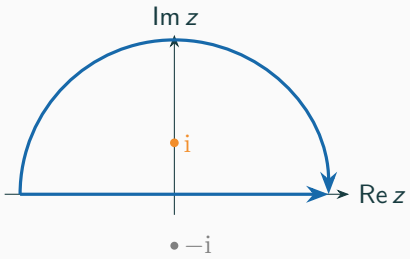
The partial derivatives are continuous; both C–R equations hold globally; and $f'(z) = 2z$ agrees with $u_x + \mathrm{i}v_x$.

Harmonicity is necessary locally; on a simply connected domain it also supports a globally defined harmonic conjugate.

Worked example 2: choose a contour before computing

Evaluate $I = \int_{-\infty}^{\infty} (x^2 + 1)^{-2}\,\mathrm{d}x$.

The integrand decays as $|z|^{-4}$, so the large upper semicircle contributes zero. Only the double pole $z = \mathrm{i}$ lies inside.



Worked example 2: residue and verification

$$\operatorname{Res}\left(\frac{1}{(z^2+1)^2};\mathrm{i}\right) = \frac{\mathrm{d}}{\mathrm{d}z}\frac{1}{(z+\mathrm{i})^2}\bigg|_{z=\mathrm{i}} = \frac{1}{4\mathrm{i}}, \qquad I = \frac{\pi}{2}.$$

Check

The integrand is positive and even; the answer is real and positive; and direct differentiation confirms the second-order residue.

Worked example 3: map a half-plane to the unit disk

Find a Möbius transformation that maps $\operatorname{Re} z > 0$ onto $|w| < 1$ and sends $z = 1$ to $w = 0$.

$$w = \frac{z-1}{z+1}.$$

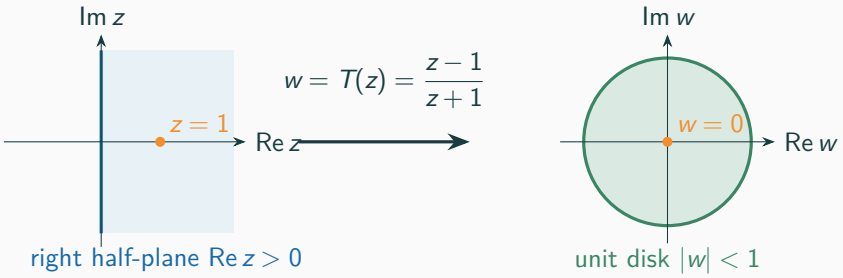
For $z = \mathrm{i}y$ on the boundary,

$$|w|^2 = \frac{|\mathrm{i}y - 1|^2}{|\mathrm{i}y + 1|^2} = \frac{y^2+1}{y^2+1} = 1.$$

Thus the imaginary axis maps to the unit circle.

Worked example 3: test the interior and inverse map

The boundary maps to $|w| = 1$; one interior test point determines the correct side.



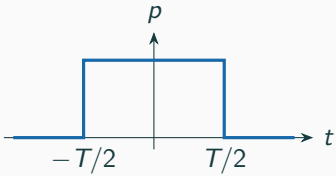
Checks

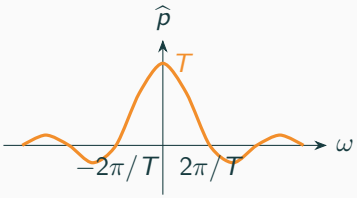
$z = iy \mapsto |w| = 1$, $z = 1 \mapsto 0$, and $z = (1 + w)/(1 - w)$ confirms one-to-one correspondence for $|w| < 1$.

Worked example 4: transform one rectangular pulse

Let $p(t) = 1$ for $|t| < T/2$ and $p(t) = 0$ otherwise ($T > 0$).
Use $\operatorname{sinc} x = \sin x / x$, with $\operatorname{sinc} 0 = 1$. Then

$$\widehat{p}(\omega) = \int_{-T/2}^{T/2} \mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t = \frac{2\sin(\omega T/2)}{\omega} = T\operatorname{sinc}\left(\frac{\omega T}{2}\right).$$



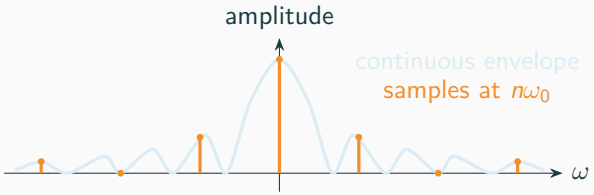


Wider pulses produce narrower spectra; a time shift changes phase, not magnitude.

Worked example 4: periodic repetition changes the tool

Repeat the pulse every $P > T$, with $\omega_0 = 2\pi/P$. The periodic train uses a Fourier series,

$$c_n = \frac{1}{P}\int_{-T/2}^{T/2} \mathrm{e}^{-\mathrm{i}n\omega_0 t}\,\mathrm{d}t = \frac{T}{P}\,\mathrm{sinc}\left(\frac{n\omega_0 T}{2}\right).$$



Method choice

Use the transform for an isolated aperiodic signal; use the series for its periodic repetition.

Worked example 5: transform an engineering IVP

For a unit-step force,

$$y'' + 2y' + 5y = u(t), \qquad y(0) = y'(0) = 0.$$

Transforming gives

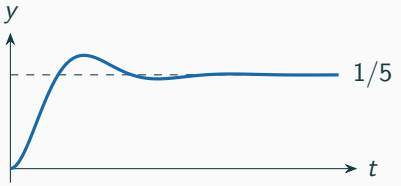
$$Y(s) = \frac{1}{s(s^2 + 2s + 5)} = \frac{1}{5s} - \frac{s+1}{5[(s+1)^2 + 4]} - \frac{1}{5[(s+1)^2 + 4]}.$$

The system poles $-1 \pm 2\mathrm{i}$ predict a decaying oscillatory transient. The input contributes the additional pole at $s = 0$ in $Y(s)$.

Worked example 5: invert, interpret and verify

$$y(t) = \frac{1}{5}\left[1 - \mathrm{e}^{-t}\cos 2t - \frac{1}{2}\mathrm{e}^{-t}\sin 2t\right] u(t).$$

- decay rate: 1;
- damped frequency: 2;
- final value: $1/5$;
- initial value and slope: zero.



Worked example 6: transform a heat boundary-value problem

For a semi-infinite rod,

$$u_t = \alpha u_{xx}, \quad x > 0, \qquad u(x,0) = 0, \quad u(0,t) = U_0, \quad u(\infty, t) = 0.$$

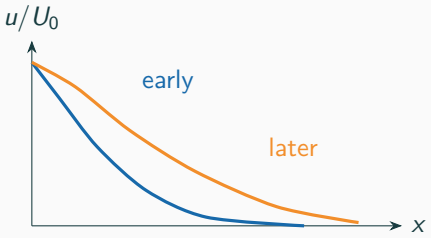
Transforming in time yields

$$\alpha U_{xx} - sU = 0, \qquad U(0,s) = \frac{U_0}{s}, \qquad U(\infty, s) = 0,$$

Choose $\operatorname{Re}\sqrt{s/\alpha} > 0$ for $\operatorname{Re} s > 0$, giving $U(x,s) = U_0 s^{-1} \exp[-x\sqrt{s/\alpha}]$.

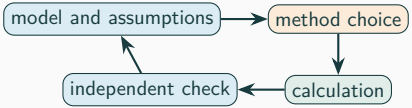
Worked example 6: inversion reveals diffusion scaling

$$u(x,t) = U_0 \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right).$$



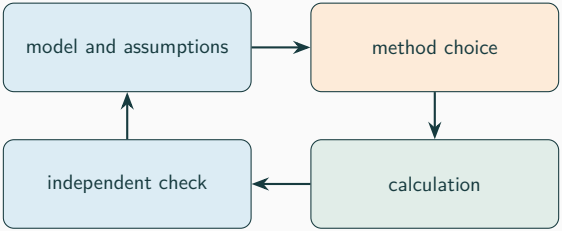
The penetration distance scales as $x \sim \sqrt{\alpha t}$. For fixed $x > 0$, $u \to 0$ as $t \to 0^+$. For $t > 0$, $u(0,t) = U_0$. The sudden boundary change creates a corner discontinuity at $(0,0)$.

Every complete solution closes a verification loop



Check domains, convergence, initial/boundary data, dimensions, limiting cases and agreement with the governing equation.

Every complete solution closes a verification loop



Check domains, convergence, initial/boundary data, dimensions, limiting cases and agreement with the governing equation.

Future Directions

Future directions and course boundaries

Direction	Connection to this course	Course status
Elliptic functions	extend complex functions to double periodicity	future study
Multivariate transforms	generalize transform kernels to several variables	future study
Fractional models	add memory through noninteger derivatives	future study
Numerical ODE/PDE methods	approximate models without closed-form transforms	separate course
Advanced control design	address robustness, synthesis and controller tuning	separate course

Boundary

These are natural extensions, not missing prerequisites and not part of the present assessment.

Choosing a method

Problem structure	Primary tool	Conditions to check
analytic function or contour	Cauchy theory, series, residues	domain, singularities, orientation
periodic signal	Fourier series	period, symmetry, jump convergence
isolated spectrum or pulse	Fourier transform	convention, integrability, bandwidth
causal transient or IVP	Laplace transform	initial data, ROC, causality
LTI or feedback system	transfer function, convolution	zero-state assumption, poles, stability
time-dependent PDE	Laplace transform in time	initial and boundary conditions

Selection rule

Choose the representation that exposes the structure while preserving every condition carried by the model.

Course summary

What the course enables

- analyze complex functions locally and globally;
- select series, contour or mapping methods;
- move between time, frequency and transform domains;
- interpret poles, modes, responses and stability.

What a complete solution states

- domain, branch, ROC and model assumptions;
- singularities, poles, zeros and boundary data;
- method choice, orientation and initial terms;
- analytic, numerical or physical verification.

Course principle

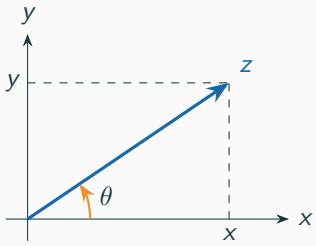
Structure before calculation; conditions before formulas; verification before interpretation.

Review Reference Appendix

Concepts, conditions, methods and quick checks for L01–L15

L01 at a glance: complex numbers — Part I

$$z = x + \mathrm{i}y = r(\cos\theta + \mathrm{i}\sin\theta) = r\mathrm{e}^{\mathrm{i}\theta}.$$



- $|z| = \sqrt{x^2 + y^2}$ and $\bar{z} = x - \mathrm{i}y$;
- $z\bar{z} = |z|^2$;
- multiplication adds arguments;
- division subtracts arguments.

L01 at a glance: complex numbers — Part II

For $z = re^{i\theta}$, the n distinct roots are

$$z_k = r^{1/n} \exp\left[\frac{i(\theta + 2k\pi)}{n}\right], \qquad k = 0, \ldots, n-1.$$

Geometry and method

The roots lie on $|z| = r^{1/n}$ with angular spacing $2\pi/n$. Choose one argument θ, generate all $k = 0, \ldots, n-1$, then verify symmetry and count.

Common errors

Omitting branches of the argument; forgetting that inequalities use modulus; confusing $\bar{z}$ with $1/z$ when $|z| \neq 1$.

L02 at a glance: analyticity — Part I

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}.$$

The limit must be independent of the direction of Δz.

For $f = u + \mathrm{i}v$,

$$u_x = v_y, \qquad u_y = -v_x, \qquad f' = u_x + \mathrm{i}v_x.$$

Continuous first partial derivatives plus C–R in a neighborhood give the usual sufficient test.

Conditions

A complex limit must agree along every path. Differentiability at one point does not imply analyticity on an open neighborhood.

L02 at a glance: analyticity — Part II

If $f = u + \mathrm{i}v$ is analytic and has continuous second derivatives, then

$$\nabla^2 u = u_{xx} + u_{yy} = 0, \qquad \nabla^2 v = v_{xx} + v_{yy} = 0.$$

Question	Diagnostic
limit exists?	compare at least two paths
complex differentiable?	test directional independence/C–R
analytic on a domain?	test every point in an open set
harmonic conjugate exists?	check harmonicity and domain topology

Check

Harmonicity is necessary for a real or imaginary part of an analytic function; a global conjugate also depends on the topology of the domain.

L03 at a glance: elementary functions — Part I

$$e^{x+iy} = e^x(\cos y + i \sin y), \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

- e^z is entire, nonzero and $2\pi i$-periodic.
- $\sin z$ and $\cos z$ are entire but unbounded in imaginary directions.
- Familiar derivative identities remain valid in the complex plane.

Hyperbolic and trigonometric functions are linked by imaginary arguments.

Use and check

Rewrite in exponentials to derive identities and locate zeros; do not import boundedness properties of real sine and cosine into the complex plane.

L03 at a glance: elementary functions — Part II

$$\log z = \ln|z| + \mathrm{i}(\arg z + 2k\pi), \qquad \operatorname{Log} z = \ln|z| + \mathrm{i}\operatorname{Arg} z.$$

On a fixed analytic branch,

$$(\operatorname{Log} z)' = \frac{1}{z}, \qquad z^{\alpha} = \exp(\alpha \operatorname{Log} z), \qquad (z^{\alpha})' = \alpha z^{\alpha-1}.$$

Common error

Algebraic identities involving logarithms can fail across incompatible branch choices. Always state the cut and the branch of Arg.

L04 at a glance: complex integration — Part I

For a piecewise smooth contour $C : z = z(t)$,

$$\int_C f(z)\,\mathrm{d}z = \int_a^b f(z(t))z'(t)\,\mathrm{d}t, \qquad \left|\int_C f\,\mathrm{d}z\right| \le ML.$$

If f has an antiderivative in the domain, the integral is path independent. If f is analytic on and inside a positively oriented simple closed contour,

$$\oint_C f(z)\,\mathrm{d}z = 0.$$

Conditions

Track orientation and regularity. Cauchy–Goursat requires analyticity throughout the relevant interior; path independence requires a suitable domain or an antiderivative.

L04 at a glance: complex integration — Part II

$$f^{(n)}(z_0) = \frac{n!}{2\pi \mathrm{i}} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z, \qquad z_0 \in \operatorname{int} C.$$

- Cauchy estimates bound derivatives by boundary values.
- Liouville: a bounded entire function is constant.
- Maximum modulus occurs on the boundary for a nonconstant analytic function.

Method and consequences

Match the denominator power to the derivative order before calculating. Cauchy estimates lead to Liouville's theorem and the maximum-modulus principle; all require the stated domain conditions.

L05 at a glance: series — Part I

$$\sum a_n (z - z_0)^n \begin{cases} \text{converges absolutely}, & |z - z_0| < R, \\ \text{diverges}, & |z - z_0| > R, \end{cases} \qquad \frac{1}{R} = \limsup_{n \to \infty} |a_n|^{1/n}.$$

Boundary points require separate tests. Inside the disk, termwise differentiation and integration preserve the radius of convergence.

$$\frac{1}{1 - z} = \sum_{n=0}^{\infty} z^n, \qquad |z| < 1.$$

Check

Ratio and root tests determine the interior radius, not boundary convergence. Record the center before comparing the distance to the nearest singularity.

L05 at a glance: series — Part II

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n.$$

If

$$f(z) = (z - z_0)^m g(z), \qquad g(z_0) \neq 0,$$

then z_0 is a zero of order m.

Quick method

Find the nearest singularity to obtain R; factor first to identify zero order; use known series before repeated differentiation.

Check

Taylor coefficients require analyticity at the center. Zero order is defined only when the remaining factor is analytic and nonzero there.

L06 at a glance: residues — Part I

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \qquad r < |z - z_0| < R.$$

Principal part	Singularity
absent	removable
finitely many negative powers	pole
infinitely many negative powers	essential

Different annuli can produce different Laurent expansions of the same formula.

Check

State the annulus before writing the series. A nonisolated singularity is not classified by this principal-part test.

L06 at a glance: residues — Part II

$$\operatorname{Res}(f, z_0) = a_{-1} = \frac{1}{2\pi \mathrm{i}} \oint f(z)\,\mathrm{d}z.$$

For a pole of order m,

$$\operatorname{Res}(f, z_0) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}} \left[(z - z_0)^m f(z) \right].$$

Prefer a limit for simple poles, a known series when available, and derivatives only when necessary.

Method check

Classify the point first, choose the shortest valid residue formula, and verify that the selected pole lies inside the positively oriented contour.

L07 at a glance: conformal mappings — Part I

If f is analytic near z_0 and $f'(z_0) \neq 0$, then locally

$$f(z_0 + \Delta z) - f(z_0) \approx f'(z_0)\Delta z.$$

Thus $|f'(z_0)|$ gives local scale and $\arg f'(z_0)$ gives local rotation. Critical points with $f'(z_0) = 0$ need separate analysis and are not conformal in the usual local one-to-one sense.

Check

Analyticity alone is insufficient: conformality at z_0 also requires $f'(z_0) \neq 0$ and a locally one-to-one interpretation.

L07 at a glance: conformal mappings — Part II

$$w = \frac{az + b}{cz + d}, \qquad ad - bc \neq 0.$$

- generalized circles map to generalized circles;
- three point correspondences determine the map;
- map the boundary first, then test one interior point;
- compositions with z^{α}, $\exp z$ or $\operatorname{Log} z$ handle wedges and strips.

Method check

Map the boundary, test one interior point, then check poles, one-to-one behavior and compatible branch choices on the proposed domain.

L08 at a glance: applications — Part I

Match contour geometry to the integrand:

- rational decay: upper/lower semicircle;
- e^{iax}: choose the half-plane where the exponential decays;
- positive-axis powers/logarithms: keyhole contour and branch cut;
- even rational functions: symmetry often halves the work.

Check

Never discard an arc before proving decay. Exclude boundary poles, state indentation contributions and keep the contour orientation consistent.

L08 at a glance: applications — Part II

If f is meromorphic inside C and has no zeros or poles on C,

$$\frac{1}{2\pi \mathrm{i}} \oint_C \frac{f'(z)}{f(z)}\,\mathrm{d}z = N_Z - N_P.$$

Rouché's theorem: if $|g| < |f|$ on C, then f and $f+g$ have the same number of zeros inside C.

Quick use

Choose the dominant term on the boundary, prove a strict inequality, then count the simpler zeros.

Conditions

The argument principle requires no zeros or poles on C; Rouché requires a strict boundary inequality on the entire contour.

L09 at a glance: Fourier analysis — Part I

For period $2L$,

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,\mathrm{d}x, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,\mathrm{d}x.$$

At a point of continuity the series converges to f; at a jump it converges to $[f(x^-) + f(x^+)]/2$ under standard piecewise smoothness conditions.

Method and check

Use parity before integrating, retain the period convention, and distinguish pointwise convergence from mean-square convergence.

L09 at a glance: Fourier analysis — Part II

With $\widehat{f}(\omega) = \int f(t)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t$,

$$f(t-t_0) \leftrightarrow \mathrm{e}^{-\mathrm{i}\omega t_0}\widehat{f}(\omega), \quad f'(t) \leftrightarrow \mathrm{i}\omega\widehat{f}(\omega), \quad f * g \leftrightarrow \widehat{f}\widehat{g}.$$

$$\int_{-\infty}^{\infty} |f(t)|^2\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\widehat{f}(\omega)|^2\,\mathrm{d}\omega.$$

Use and check

A series describes periodic data; a transform describes an aperiodic signal. Keep the course normalization fixed and verify symmetry, signs and integrability assumptions.

L10 at a glance: Laplace foundations — Part I

The unilateral transform exists in a right half-plane when f is piecewise continuous on finite intervals and of exponential order:

$$|f(t)| \le M\mathrm{e}^{at} \quad (t \ge T) \quad \Longrightarrow \quad F(s) \text{ converges for } \operatorname{Re} s > a.$$

Functions such as e^{t^2} can grow too rapidly; an internal nonintegrable singularity can also prevent existence.

Check

State whether the transform is unilateral or bilateral. The algebraic formula and its ROC jointly identify the signal class.

L10 at a glance: Laplace foundations — Part II

$$1 \leftrightarrow \frac{1}{s}, \quad e^{at} \leftrightarrow \frac{1}{s-a}, \quad \sin \omega t \leftrightarrow \frac{\omega}{s^2+\omega^2}, \quad t^n \leftrightarrow \frac{n!}{s^{n+1}}.$$

- left-half-plane poles produce decaying modes;
- imaginary-axis poles produce persistent oscillation;
- right-half-plane poles produce growing modes.

Method check

Pole conclusions assume no hidden cancellation and the appropriate ROC. Complex-conjugate poles are required for real oscillatory signals.

L11 at a glance: Laplace properties — Part I

$$e^{at}f(t) \leftrightarrow F(s-a), \quad u(t-a)f(t-a) \leftrightarrow e^{-as}F(s),$$
$$f^{(n)}(t) \leftrightarrow s^n F(s) - s^{n-1}f(0^-) - \cdots - f^{(n-1)}(0^-).$$

For a causal periodic function of period T,

$$F(s) = \frac{\int_0^T e^{-st}f(t)\,dt}{1 - e^{-sT}}.$$

Check

Distinguish a time delay from exponential weighting, retain every initial term, and integrate exactly one fundamental period.

L11 at a glance: Laplace properties — Part II

$$(f * g)(t) = \int_0^t f(\tau) g(t-\tau)\,\mathrm{d}\tau, \qquad \mathcal{L}\{f * g\} = FG.$$

- step $u(t-a)$ switches a persistent input;
- a pulse is the difference of two steps;
- $\delta(t-a)$ models a concentrated impulse;
- convolution accumulates shifted impulse responses.

Method check

Use 0^- when impulses at the origin are included. Apply convolution when a useful product of transforms is visible and interpret it as accumulated impulse response.

L12 at a glance: inverse transforms — Part I

1. simplify and make a rational function proper;
2. identify shifts, scaling and standard quadratic forms;
3. use partial fractions for rational transforms;
4. use convolution for useful products;
5. use Bromwich/residues when analytic structure is essential.

Check

Verify by a forward transform, initial/final values only where their hypotheses hold, causal support and dimensional consistency.

L12 at a glance: inverse transforms — Part II

$$f(t) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} \mathrm{e}^{st} F(s)\, \mathrm{d}s,$$

where the line lies in the ROC and to the right of all singularities for a causal transform.

- poles contribute residues;
- repeated poles produce polynomial factors in t;
- branch points require a branch cut and cut contribution;
- improper rational parts may generate impulse derivatives.

Method check

Place the Bromwich line in the ROC, choose closure from the sign of time, and treat branch-cut contributions separately from pole residues.

L13 at a glance: ODE initial-value problems — Part I

$$\text{ODE} + \text{initial data} \xrightarrow{\mathcal{L}} \text{algebraic equation for } Y(s) \xrightarrow{\mathcal{L}^{-1}} y(t).$$

1. encode switching/impulses before transforming;
2. keep every initial-value term;
3. solve for $Y(s)$ and expose poles;
4. invert and check the ODE, data and limiting behavior.

Check

Confirm the recovered solution satisfies the differential equation, every initial condition, switching or jump conditions and the expected long-time behavior.

L13 at a glance: ODE initial-value problems — Part II

Model	Differential equation	Interpretation
mass–spring–damper	$my'' + cy' + ky = F$	damping/resonance
RLC circuit	$Lq'' + Rq' + q/C = E$	charge/current response
coupled system	$\mathbf{x}' = A\mathbf{x} + B\mathbf{u}$	modes/eigenvectors
beam in x	spatial ODE + boundary data	deflection profile

Method check

Poles predict available modes, but initial and forcing terms determine which modes appear. Preserve units and distinguish zero-input from zero-state response.

L14 at a glance: systems and PDEs — Part I

$$H(s) = \frac{Y(s)}{X(s)} \quad \text{under zero initial conditions,} \qquad y = h * x.$$

Cascading two systems multiplies transfer functions and convolves impulse responses. A convolution-type Volterra equation

$$y(t) = f(t) + \int_0^t k(t-\tau)y(\tau)\,\mathrm{d}\tau$$

becomes $Y = F + KY$, hence $Y = F/(1-K)$ where defined.

Check

Transfer functions assume zero initial conditions. Check causality, stability and the region where algebraic division by $1 - K(s)$ is valid.

L14 at a glance: systems and PDEs — Part II

$$\text{PDE in } (x,t) \xrightarrow{\mathcal{L}_t} \text{ODE in } x \xrightarrow{\text{boundary data}} U(x,s) \xrightarrow{\mathcal{L}^{-1}} u(x,t).$$

Heat diffusion produces exponential spatial decay in $\sqrt{s}$; wave problems retain oscillatory propagation structure.

Common error

Transforming in time does not remove spatial boundary conditions or the initial terms generated by time derivatives.

Interpretation

Diffusion smooths and attenuates spatial variation; wave propagation transports oscillatory structure. Select the inverse transform consistent with the transformed boundary-value problem.

L15 at a glance: control and stability — Part I

For negative feedback,

$$T(s) = \frac{G(s)}{1 + G(s)H_f(s)}, \qquad 1 + G(s)H_f(s) = 0$$

is the closed-loop characteristic equation.

- first-order time constant: $\tau = 1/|p|$;
- second-order damping ratio ζ controls overshoot;
- natural frequency ω_n sets the response scale;
- Bode magnitude and phase summarize sinusoidal steady response.

Check

Separate open-loop poles from roots of the closed-loop characteristic equation. Response metrics assume the stated standard first- or second-order model.

L15 at a glance: control and stability — Part II

For $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, equilibria satisfy $\mathbf{f}(\mathbf{x}_e) = 0$. Linearization uses

$$\dot{\xi} = A\xi, \qquad A = D\mathbf{f}(\mathbf{x}_e).$$

Negative real parts imply local asymptotic stability for a hyperbolic equilibrium; a positive real part implies instability. Saddles are unstable, centers need nonlinear care, and nonhyperbolic equilibria cannot be decided by linearization alone.

Method check

Find equilibria first, form the Jacobian at each equilibrium, classify its eigenvalues, and state explicitly when the linear test is inconclusive.

Formula reference I: complex numbers and roots

$$z = x + \mathrm{i}y = r\mathrm{e}^{\mathrm{i}\theta}, \qquad \bar{z} = x - \mathrm{i}y,$$

$$|z|^2 = z\bar{z}, \qquad \frac{1}{z} = \frac{\bar{z}}{|z|^2} \quad (z \neq 0),$$

$$|z_1 z_2| = |z_1||z_2|,$$

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2 \pmod{2\pi},$$

$$(\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta.$$

$$z_k = r^{1/n} \exp\left[\frac{\mathrm{i}(\theta + 2k\pi)}{n}\right], \quad k = 0, \ldots, n-1.$$

$$|z_1 + z_2| \leq |z_1| + |z_2|,$$

$$\big||z_1| - |z_2|\big| \leq |z_1 - z_2|.$$

Check

Keep every root branch; distinguish $\arg z$ from the principal value $\operatorname{Arg} z$.

Formula reference II: derivatives, analyticity and harmonicity

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z},$$

$$u_x = v_y, \qquad u_y = -v_x,$$

$$f'(z) = u_x + \mathrm{i} v_x = v_y - \mathrm{i} u_y,$$

$$\nabla^2 u = u_{xx} + u_{yy} = 0, \qquad \nabla^2 v = 0.$$

C–R is necessary wherever $f = u + \mathrm{i}v$ is differentiable.

C–R plus continuous first partials in a neighborhood is the standard practical sufficiency test.

Check

Differentiability at one point is not analyticity. Global conjugates also require suitable domain topology.

Formula reference III: elementary complex functions

$$e^{x+iy} = e^x(\cos y + i \sin y),$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2},$$

$$\cosh z = \frac{e^{z} + e^{-z}}{2}, \qquad \sinh z = \frac{e^{z} - e^{-z}}{2}.$$

$$\log z = \ln|z| + i(\arg z + 2k\pi),$$

$$\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z,$$

$$z^{\alpha} = \exp(\alpha \operatorname{Log} z),$$

$$(\operatorname{Log} z)' = \frac{1}{z}, \qquad (z^{\alpha})' = \alpha z^{\alpha-1}.$$

Check

Derivative identities for $\operatorname{Log} z$ and z^{α} apply only on a fixed analytic branch; state the cut and excluded points.

Formula reference IV: contours and Cauchy theory

$$\int_C f(z)\,\mathrm{d}z = \int_a^b f(z(t))z'(t)\,\mathrm{d}t,$$

$$\left|\int_C f(z)\,\mathrm{d}z\right| \le ML, \qquad M = \max_C |f|, \quad L = \text{length}(C),$$

$$\int_{z_a}^{z_b} f(z)\,\mathrm{d}z = F(z_b) - F(z_a) \qquad (F' = f),$$

$$\oint_C f(z)\,\mathrm{d}z = 0,$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi \mathrm{i}} \oint_C \frac{f(z)}{(z - z_0)^{n+1}}\,\mathrm{d}z.$$

Conditions and checks

Track orientation and limits. For the closed-contour identities, C is positively oriented and simple, and f is analytic on a neighborhood of C and its interior. Also require z_0 inside C.

Formula reference V: power, Taylor and Laurent series

$$\frac{1}{1-w} = \sum_{n=0}^{\infty} w^n, \qquad |w| < 1,$$

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \qquad \frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n},$$

$$a_n = \frac{f^{(n)}(z_0)}{n!}, \qquad |z - z_0| < R,$$

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \qquad r < |z - z_0| < R,$$

$$a_n = \frac{1}{2\pi \mathrm{i}} \oint_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, \mathrm{d}\zeta.$$

Check

State the disk or annulus and test its boundary separately; Laurent forms can change from one annulus to another.

Formula reference VI: singularities, residues and counting

$$\operatorname{Res}(f, z_0) = a_{-1},$$

$$\operatorname{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z),$$

$$\operatorname{Res}(f, z_0) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}} \left[(z - z_0)^m f(z) \right].$$

$$\oint_C f \,\mathrm{d}z = 2\pi \mathrm{i} \sum \operatorname{Res}(f, z_k),$$

$$N_Z - N_P = \frac{1}{2\pi \mathrm{i}} \oint_C \frac{f'}{f} \,\mathrm{d}z,$$

$$\operatorname{Res}(f, \infty) = -\sum_{z_k \in \mathbb{C}} \operatorname{Res}(f, z_k).$$

Check

The limit formula assumes a simple pole; the derivative formula assumes order m. Counting uses a positive simple contour with no boundary zeros/poles. The infinity identity assumes meromorphicity on the extended plane.

Formula reference VII: mappings and real integrals

$$w = \frac{az+b}{cz+d}, \qquad ad-bc \neq 0, \qquad w'(z) = \frac{ad-bc}{(cz+d)^2}, \qquad z = \frac{dw-b}{a-cw}.$$

Möbius maps preserve generalized circles. Test the boundary and one interior point.

$$\int_{-\infty}^{\infty} R(x)\,\mathrm{d}x = 2\pi\mathrm{i} \sum_{\operatorname{Im} z_k > 0} \operatorname{Res}(R; z_k).$$

Conditions

The real-integral formula assumes rational R with no real poles and $R(z) = O(|z|^{-2})$ at infinity. Use a positively oriented upper semicircle. Boundary poles require principal values and indentation contributions.

Formula reference VII continued: principal values

For a possible singularity at 0,

$$\mathrm{PV}\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \lim_{R\to\infty}\lim_{\varepsilon\to 0^+}\left[\int_{-R}^{-\varepsilon} f(x)\,\mathrm{d}x + \int_{\varepsilon}^{R} f(x)\,\mathrm{d}x\right],$$

provided the stated limits exist.

Contour calculation

Write the closed-contour integral as real segments, the large arc and each small indentation. Apply the residue theorem to this entire contour, then isolate the real segments.

Check

The principal value can exist even when the ordinary improper integral fails. Do not discard small-arc contributions at boundary poles.

Formula reference VIII: Fourier series

For a function of period $2L$,

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi t}{L} + b_n \sin \frac{n\pi t}{L} \right),$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(t) \cos \frac{n\pi t}{L} \, \mathrm{d}t, \qquad b_n = \frac{1}{L} \int_{-L}^{L} f(t) \sin \frac{n\pi t}{L} \, \mathrm{d}t,$$

$$c_n = \frac{1}{2L} \int_{-L}^{L} f(t) \mathrm{e}^{-\mathrm{i}n\pi t/L} \, \mathrm{d}t, \qquad f(t) \sim \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{\mathrm{i}n\pi t/L},$$

$$\frac{1}{L} \int_{-L}^{L} |f|^2 \, \mathrm{d}t = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

Check

Use symmetry first. Under the usual Dirichlet conditions, a jump converges to the midpoint of its one-sided limits.

Formula reference IX: Fourier transform

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} f(t) \mathrm{e}^{-\mathrm{i}\omega t} \, \mathrm{d}t, \qquad f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(\omega) \mathrm{e}^{\mathrm{i}\omega t} \, \mathrm{d}\omega,$$

$$f(t - t_0) \longleftrightarrow \mathrm{e}^{-\mathrm{i}\omega t_0} \widehat{f}(\omega), \qquad f(at) \longleftrightarrow \frac{1}{|a|} \widehat{f}(\omega/a),$$

$$f^{(n)}(t) \longleftrightarrow (\mathrm{i}\omega)^n \widehat{f}(\omega), \qquad t^n f(t) \longleftrightarrow \mathrm{i}^n \frac{\mathrm{d}^n \widehat{f}}{\mathrm{d}\omega^n},$$

$$f * g \longleftrightarrow \widehat{f}\widehat{g}, \qquad \int |f|^2 \, \mathrm{d}t = \frac{1}{2\pi} \int |\widehat{f}|^2 \, \mathrm{d}\omega.$$

Check

Use $a \neq 0$ in scaling. For classical inversion, a sufficient condition is $f, \widehat{f} \in L^1$ at continuity points. Derivative identities require integrable derivatives and vanishing boundary terms; Parseval uses L^2.

Formula reference X: Laplace foundations and pairs

$$F(s) = \int_{0^-}^{\infty} \mathrm{e}^{-st} f(t)\,\mathrm{d}t, \qquad s = \sigma + \mathrm{i}\omega.$$

Piecewise continuity and exponential order provide the standard practical existence test.

$$1 \leftrightarrow \frac{1}{s}, \qquad t^n \leftrightarrow \frac{n!}{s^{n+1}}, \qquad \mathrm{e}^{at} \leftrightarrow \frac{1}{s-a}.$$

$$\sin \omega t \leftrightarrow \frac{\omega}{s^2+\omega^2},$$
$$\cos \omega t \leftrightarrow \frac{s}{s^2+\omega^2},$$
$$\sinh at \leftrightarrow \frac{a}{s^2-a^2},$$
$$\cosh at \leftrightarrow \frac{s}{s^2-a^2}.$$

Check

These pairs are causal. For real a, ω, use $\operatorname{Re} s > 0$ for powers and trigonometric pairs, $\operatorname{Re} s > a$ for e^{at}, and $\operatorname{Re} s > |a|$ for hyperbolic pairs.

Formula reference XI: Laplace operational properties

$$\mathcal{L}\{af+bg\} = aF + bG, \qquad \mathcal{L}\{f(at)\} = \frac{1}{a}F(s/a) \quad (a>0),$$

$$\mathcal{L}\{\mathrm{e}^{at}f(t)\} = F(s-a), \qquad \mathcal{L}\{u(t-a)f(t-a)\} = \mathrm{e}^{-as}F(s),$$

$$\mathcal{L}\{f^{(n)}\} = s^nF - \sum_{k=0}^{n-1} s^{n-1-k}f^{(k)}(0^-),$$

$$\mathcal{L}\{t^nf(t)\} = (-1)^nF^{(n)}(s), \qquad \mathcal{L}\{f*g\} = F(s)G(s),$$

$$\mathcal{L}\{f_{\mathrm{per}}\} = \frac{\int_0^T \mathrm{e}^{-st}f(t)\,\mathrm{d}t}{1-\mathrm{e}^{-sT}}.$$

Check

For a delay use $a \geq 0$. Convolution is causal and the periodic formula uses the causal restriction of a waveform. Retain initial terms and the common ROC.

Formula reference XII: inversion and differential equations

$$f(t) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} \mathrm{e}^{st} F(s)\,\mathrm{d}s,$$

$$f(t) = \sum_{k} \operatorname{Res}\bigl(\mathrm{e}^{st} F(s); s_k\bigr) \quad (t > 0),$$

$$\mathcal{L}\{y''\} = s^2 Y - s y(0^-) - y'(0^-),$$

$$u_t = \alpha u_{xx} \Longrightarrow sU - u(x,0) = \alpha U_{xx},$$

$$u_{tt} = c^2 u_{xx} \Longrightarrow s^2 U - s u(x,0) - u_t(x,0) = c^2 U_{xx}.$$

Check

The residue sum assumes finitely many poles and a vanishing closing contour, with no branch-cut contribution or impulse terms. Put γ in the ROC. Preserve the PDE initial and boundary data.

Formula reference XIII: systems, response and stability

$$H(s) = \frac{Y(s)}{X(s)} \quad \text{(zero ICs)},$$

$$y = h * x,$$

$$T(s) = \frac{G(s)}{1 + G(s)H_f(s)},$$

$$S(s) = \frac{1}{1 + G(s)H_f(s)},$$

$$G_2(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

$$M_p = \mathrm{e}^{-\pi\zeta/\sqrt{1-\zeta^2}},$$

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}},$$

$$t_s \approx \frac{4}{\zeta\omega_n},$$

$$\dot{\boldsymbol{\xi}} = A\boldsymbol{\xi},$$

$$\det(\lambda I - A) = 0.$$

Check

For a standard second-order unit-step response, $0 < \zeta < 1$ and M_p is a ratio. t_s is the approximate 2% settling time. Eigenvalues with negative real parts imply local asymptotic stability; a positive real part implies instability.

$$e^{i\pi} + 1 = 0$$